

A class of Calabi-Yau threefolds as manifolds of $SU(2)$ structure

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Abstract

A class of abelian fibered Calabi-Yau threefolds $\mathcal{X}_{m,n}$ is shown yield $SU(2)$ structure, in addition to the standard $SU(3)$ holonomy. Compactification of type II string theory on a manifold in this class give a 4D effective supergravity theory in which the topology spontaneously breaks $\mathcal{N} = 4$ to $\mathcal{N} = 2$ supersymmetry. The breaking occurs at a scale hierarchically lower than the compactification scale when the \mathbb{P}^1 base is large compared to the T^4 fiber. We analyze the moduli space of $SU(2)$ structure metrics of the $\mathcal{N} = 4$ theory and its restriction to the moduli space of Calabi-Yau metrics of the $\mathcal{N} = 2$ theory, showing that the latter agrees with the expectation computed from triple intersection numbers in the classical limit. Finally, we analyze the twisted cohomology ring associated with the $SU(2)$ structure of $\mathcal{X}_{m,n}$ and show that the breaking of $\mathcal{N} = 4$ to $\mathcal{N} = 2$ is conveniently summarized in the lifting cohomology classes as one passes to the standard cohomology ring, with massive modes persisting as torsion classes when the coupling is nonminimal. The analysis is facilitated by the existence of explicit first-order metrics obtained by classical supergravity dualities.

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1 Introduction

Calabi-Yau manifolds are the bread and butter of string theory compactifications. From traditional heterotic models, to type IIB flux compactifications, to IIA intersecting brane models based on Calabi-Yau orientifolds, it is hard to imagine model building that is not in some way related to Calabi-Yau compactifications.¹ The study of Calabi-Yau mirror symmetry in the late 80s and early 1990s ushered in a renaissance of cross fertilization between mathematics and physics that continues to yield new insights, and has brought us the inspiring edifice of topological string theory in its closed and open forms. Given that Calabi-Yau manifolds have been studied for more than two and a half decades [3], it might seem unlikely that any novel low hanging fruit would not already have been plucked.

The rise of flux compactifications a decade ago brought one such development. It was once thought that nonzero flux was forbidden on energetic grounds, but we now know that that argument does not apply, due to the existence of α' corrections and classically negative tension objects such as orientifold planes and wrapped D-branes that are well defined in the microscopic theory. The flux can be thought of as a discrete choice of data that *gauges* the low energy supergravity theory, coupling vectors and scalars that would otherwise not interact. It was in this context that the first (metastable) de Sitter vacua [27] and the first compactifications with all moduli stabilized [10] were constructed. In another departure from purely geometric Calabi-Yau compactification, discrete modifications of Calabi-Yau topology were considered, which give rise to manifolds of $SU(3)$ structure rather than $SU(3)$ holonomy. (For one of many examples, see [38].) Again, the discrete data gauges the low energy supergravity theory. The general framework giving rise to 4D $\mathcal{N} = 2$ gauged supergravity in type II was described in Refs. [20, 21]. Such compactifications are in general nongeometric. When a local geometric description exists, the local compactification manifold X seems to be one of with generalized tangent bundle $(T+T^*)X$ of $SU(3) \times SU(3)$ structure.

In this paper, we return from these exotic developments to the familiar, purely geometric $\mathcal{N} = 2$ Calabi-Yau compactification of type II string theory in the absence of flux, and ask whether there might be a gauged supergravity of higher supersymmetry lurking the Calabi-Yau compactification itself. Indeed, a potential mechanism for this was sketched in Ref. [12]. It is possible that in addition to the standard $SU(3)$ holonomy of a Calabi-Yau threefold, which gives rise to a covariantly constant spinor and low energy $\mathcal{N} = 2$ supersymmetry of type II, there might be a richer $SU(2)$ structure, with twice the number of global spinors and low energy gauged $\mathcal{N} = 4$ supergravity. In this context, only one spinor would be covariantly conserved with respect to the metric connection, and the topology of the Calabi-Yau would break the $\mathcal{N} = 4$ to $\mathcal{N} = 2$ at sufficiently low energies.

We know of at least one context, where this *must* be the case. A simple model embodying many of the features of more realistic flux compactifications is the type IIB T^6/\mathbb{Z}_2 orientifold. $\mathcal{N} = 2$ vacua of this model were studied in Refs. [26, 32, 33], and it was shown in Ref. [34] that these vacua are dual to standard Calabi-Yau compactifications of type IIA string theory.

¹ G_2 compactifications are no exception. They give rise to type IIA intersecting D6 brane models on Calabi-Yau orientifolds in the weak coupling limit, and to heterotic compactifications in the case that the G_2 manifold admits a K3 fibration.

The fluxes of these $\mathcal{N} = 2$ vacua are parametrized by two integers m and n determining the NSNS and RR flux in type IIB, related to the number M of D-branes by $M = 16 - 4mn$. The IIA Calabi-Yau duals $\mathcal{X}_{m,n}$ are Abelian surface fibrations (T^4 analogs of elliptic fibrations), with m, n determining the *topology* of the fibration over the \mathbb{P}^1 base. Since the fluxes spontaneously break $\mathcal{N} = 4$ to $\mathcal{N} = 2$ in the type IIB dual, it must be the case that the topology of the $\mathcal{X}_{m,n}$ spontaneously breaks $\mathcal{N} = 4$ to $\mathcal{N} = 2$ in the Calabi-Yau compactification. Since $\mathcal{N} = 4$ requires twice the number of spinors as $\mathcal{N} = 2$, the spinor bilinears yield a richer set of tensors on $\mathcal{X}_{m,n}$ than the Kähler form and holomorphic 3-form J determined by its Calabi-Yau structure alone. We have an $SU(2)$ structure on $\mathcal{X}_{m,n}$ and a corresponding moduli space of $SU(2)$ structure metrics, of which the Calabi-Yau metrics form a small subset.

The defining feature of $SU(2)$ structure is restricted $SU(2)$ structure group of the frame bundle of $\mathcal{X}_{m,n}$, which need not (and does not) coincide with the Riemannian holonomy group from the metric connection. Instead it corresponds to a connection with torsion, and with this torsion comes a twisted differential operator, from which one can compute a twisted cohomology ring. Just as the massless fields of a Calabi-Yau compactification are intimately connected with the harmonic forms in one-to-one correspondence the ordinary de Rham cohomology, the light fields of the $\mathcal{N} = 4$ theory are connected in the same way with twisted harmonic forms and twisted cohomology. A subset of these light $\mathcal{N} = 4$ fields are the exact massless fields of the $\mathcal{N} = 2$ theory, and likewise, a subset of the twisted cohomology representatives are represent the small nontwisted cohomology ring.

When nonminimal a flux is chosen in type IIB, some of the $U(1)$ gauge groups are coupled in such a way that all charged matter has N times the minimal unit of charge, and the resulting Higgs mechanism leaves a \mathbb{Z}_N subgroup of the original $U(1)$ unbroken. Correspondingly, when nonminimal fibration data $(m, n) \neq (1, 1)$ is chosen in defining the topology of $\mathcal{X}_{m,n}$, a discrete subset of the larger twisted cohomology ring persists in the ordinary cohomology ring as torsion classes.

In this paper, we study the $SU(2)$ structure and Calabi-Yau geometry of the family $\mathcal{X}_{m,n}$ utilizing an approximate model for the metric obtained by duality to the tree level supergravity description of the type IIB T^6/\mathbb{Z}_2 orientifold with $\mathcal{N} = 2$ flux [34]. (For similar first-order analyses in the context of K3 and F-theory, see Refs. [35, 23].) This approximate description is not the only one available. Exact constructions were given in Ref. [12]. These results and their extensions will be utilized in this paper as well. However, the benefit of the approximate description is that the metric, twisted and nontwisted cohomology ring and harmonic forms, are substantially more accessible than they would be using the tools of algebraic geometry, and nevertheless suffice to yield the discrete topological data that one would want in the exact description.

Prior work on $SU(2)$ structure compactifications include a chain of interesting investigations [18, 2, 31, 39, 29, 7, 37, 6], which were primarily concerned with the low energy effective field theory as opposed to explicit examples of manifolds of $SU(2)$ structure. In contrast, we will focus on the geometry and differential topology of $SU(2)$ structure, in the explicit context of the family $\mathcal{X}_{m,n}$, touching on effective field theory only to the extent that

it relates to moduli spaces and the twisted versus ordinary cohomology. The observation that a Calabi-Yau manifold might furnish a manifold of $SU(2)$ structure was made independently by Kashani-Poor, Minasian, and Triendl, who presented preliminary results of their work at Strings 2011. Their work [40, 28] considers the low energy effective field theory from compactification on the Voisin-Borcea Calabi-Yau threefold $(K3 \times T^2)/\mathbb{Z}_2$ fibered by Enriques surfaces and familiar from Ref. [15]. These authors also study certain generalities that the present work does not, for example noting that the Euler characteristic must vanish as a condition for the existence of the global 1-forms of $SU(2)$ structure in 6D.

An outline of the paper is as follows.

In Sec. 2 we introduce notation, and discuss $SU(2)$ and $SU(3)$ structure generalities, for example, the relation between global nowhere-vanishing spinors and invariant tensors constructed from these spinors. In the case of $SU(3)$ structure we obtain the familiar J and Ω , while in the the $SU(2)$ structure the coframe bundle splits into 2D and 4D subbundles $\mathcal{F}^* = \mathcal{F}_4^* + \mathcal{F}_2^*$, with a pair of global nowhere-vanishing 1-forms trivializing a 2D subbundle of the frame bundle, together with an almost hypercomplex structure of the 4D subbundle. The torsion and twisted exterior derivatives are introduced in this section. We close the section with a discussion of the case of interest: simultaneous $SU(3)$ holonomy and $SU(2)$ structure.

Sec. 3 gives a quick review of the properties of the abelian surface fibered Calabi-Yau threefolds $\mathcal{X}_{m,n}$, and their origin via duality from the type IIB T^6/\mathbb{Z}_2 orientifold with $\mathcal{N} = 2$ flux [34]. We also sketch some of the features of the explicit constructions of Ref. [12].

Sec. 4 focuses on $\mathcal{N} = 4$ description, in which the $\mathcal{X}_{m,n}$ viewed as manifolds of $SU(2)$ structure. We discuss the space of approximate $SU(2)$ structure metrics obtained by duality from the $\mathcal{N} = 4$ theory, truncating to the tree level type IIB supergravity description in the T^6/\mathbb{Z}_2 dual. We describe the frame, harmonic forms, and moduli spaces of almost hypercomplex structures on $\mathcal{F}_4^* \mathcal{X}_{m,n}$ and almost complex structures on $\mathcal{F}_2^* \mathcal{X}_{m,n}$. This section also defines the specific torsion and twisted exterior derivatives for $\mathcal{X}_{m,n}$.

Sec. 5 turns to the lower energy theory in which $\mathcal{N} = 4$ is spontaneously broken to $\mathcal{N} = 2$, and the remaining massless degrees of freedom coincide with the standard description of the $\mathcal{X}_{m,n}$. We give the restriction of the metric from the more general $SU(2)$ form described in the previous section, and write down the Kähler form and holomorphic 3-form. Restricting the $\mathcal{N} = 4$ moduli space to the exact moduli of the $\mathcal{N} = 2$ Calabi-Yau metric, we obtain an expression for the (self-mirror) moduli space metric on the space of first-order Calabi-Yau metrics. This metric agrees with the classical (i.e., cubic prepotential) Calabi-Yau metrics computed from the classical triple intersection form on $\mathcal{X}_{m,n}$, after a field redefinition. The details of the field redefinition and metric equivalence can be found in App. A. We close this section with a discussion of harmonic forms, and define a moduli independent basis of $H_2(\mathcal{X}_{m,n}, \mathbb{R})$, which we identify with a basis from the exact description of $\mathcal{X}_{m,n}$ later in Sec. 6.4.

This brings us to Sec. 6, which relatively short, and perhaps the most novel part of the paper. In a basis defined in Secs. 4 and 5, we compute the twisted and ordinary cohomology rings of $\mathcal{X}_{m,n}$. The latter can be computed within the former, and lifts a number of classes

which become nonclosed or exact with respect to the ordinary exterior derivatives. When the topological data is nonminimal, $(m, n) \neq (1, 1)$, we find that remnants of the lifted cohomology classes persist as torsion classes. In the final part of this section, we discuss the free part of the integer homology from the point of view of explicit constructions, in order to put earlier results on firmer ground and make precise statements about how the moduli independent forms defined at the end of Sec. 5 relate to integer homology classes. In App. B we compute the double and triple intersections of divisors, extending the results of Ref. [12].

Finally, in Sec. 7, we conclude, and discuss relations to ongoing and future work.

2 SU(3) and SU(2) structure generalities

Consider a six dimensional oriented Riemannian manifold (\mathcal{X}, g) with vanishing second Stiefel-Whitney class $w_2 \in H^2(\mathcal{X}, \mathbb{Z}_2)$, so that \mathcal{X} admits a spin structure.² Let x^m denote coordinates on \mathcal{X} and let $e^{\hat{m}}_n$ be a vielbein for g_{mn} ,

$$ds^2 = g_{mn} dy^m dy^n = \delta_{\hat{m}\hat{n}} e^{\hat{m}} e^{\hat{n}}, \quad \text{where} \quad e^a = e^a_m dy^m. \quad (1)$$

The Clifford algebra is

$$\{\gamma_m, \gamma_n\} = 2g_{mn}, \quad (2)$$

where $\gamma_m = \gamma_{\hat{n}} e^{\hat{n}}_m$, and the $\gamma_{\hat{n}}$ are constant gamma matrices satisfying $\{\gamma_{\hat{m}}, \gamma_{\hat{n}}\} = 2\delta_{\hat{m}\hat{n}}$. Since \mathcal{X} is oriented, we can define a volume form and chirality operator by

$$\text{Vol}_{(6)} = e^{\hat{1}} \wedge e^{\hat{2}} \wedge e^{\hat{3}} \wedge e^{\hat{4}} \wedge e^{\hat{5}} \wedge e^{\hat{6}} = \sqrt{g} dy^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6, \quad (3)$$

$$\gamma_{(6)} = \frac{1}{6!} \epsilon_{\hat{m}\hat{n}\hat{p}\hat{q}\hat{r}\hat{s}} \gamma^{\hat{m}\hat{n}\hat{p}\hat{q}\hat{r}\hat{s}} = \frac{1}{6!} (\text{Vol}_{(6)})_{mnpqrs} \gamma^{mnpqrs}, \quad (4)$$

where gamma matrices with multiple indices denotes antisymmetrized products of gamma matrices: for example, $\gamma_{mn} = \gamma_{[m} \gamma_{n]} = \frac{1}{2}(\gamma_m \gamma_n - \gamma_n \gamma_m)$.

2.1 SU(3) structure in 6D

2.1.1 Spinors and SU(3) invariant tensors

Suppose further that there exists a global nowhere-vanishing spinor χ of positive chirality, $\gamma \chi = \chi$, which we assume without loss of generality to be normalized, $\chi^\dagger \chi = 1$. Then, \mathcal{X} has SU(3) structure.³ That is, the structure group of the frame bundle of \mathcal{X} is reduced from $SO(6)$ to SU(3), and χ determines SU(3) invariant tensors

$$J_m{}^n = i \chi^\dagger \gamma_m \chi, \quad (5)$$

$$\Omega_{mnp} = \chi^\dagger \gamma_{mnp} \chi^*. \quad (6)$$

²The spin structures are in one-to-one correspondence with elements of $H^1(\mathcal{X}, \mathbb{Z}_2)$, so there is a unique spin structure for simply connected \mathcal{X} , but may be more than one otherwise.

³A d -dimensional manifold \mathcal{X} is said to have G -structure when the structure group of the frame bundle is reduced from $SO(d)$ to a subgroup G .

In writing the latter equation, we employ a Majorana convention, in which the γ_m are imaginary and Hermitian. We assume this convention throughout the paper. From Fierz identities, it can be shown that

$$J_m{}^n J_n{}^p = -\delta_m{}^p, \quad (7)$$

$$\frac{1}{3!} J \wedge J \wedge J = \frac{i}{8} \Omega \wedge \bar{\Omega} = \text{Vol}_{(6)}. \quad (8)$$

Here, the first line implies that $J_m{}^n$ defines an almost complex structure (ACS). In the second line, the fundamental form $J = \frac{1}{2} J_{mn} dx^m \wedge dx^n$ is obtained by using the metric to lower an index of the ACS, $J_{mn} = J_m{}^p \delta_{pn}$. Given a holomorphic (antiholomorphic) frame e^j ($e^{\bar{j}}$), we have

$$J_j{}^k = i\delta_j{}^k, \quad J_{\bar{j}}{}^{\bar{k}} = -i\delta_{\bar{j}}{}^{\bar{k}}, \quad (9)$$

from which the gamma matrices $\gamma^{\bar{j}}, \gamma_k$ annihilate χ , and the gamma matrices $\gamma^j, \gamma_{\bar{k}}$ act as raising operators. Thus, we have an isomorphism between holomorphic differential forms $\omega_{i_1 \dots i_p}$ and spinors $\omega_{i_1 \dots i_p} \gamma^{i_1 \dots i_p} \chi$. The normalized negative chirality spinor χ^* satisfies $\gamma_{ijk} \chi^* = \Omega_{ijk} \chi$.

2.1.2 Torsion, contorsion, and twisted exterior derivative

When χ is covariantly constant with respect to the standard spin connection ∇ , it is straightforward to show that dJ and $d\Omega$ vanish. Then, the manifold is Calabi-Yau and the standard connection has $\text{SU}(3)$ holonomy. For more general $\text{SU}(3)$ structure manifolds, χ is conserved by a connection $\nabla^{(T)}$ with torsion,

$$\nabla_m^{(T)} \chi = (\nabla_m - \frac{1}{4} \kappa_{mnp} \gamma^{np}) \chi = 0, \quad (10)$$

where κ_{mnp} is the contorsion [24].⁴ The contorsion tensor gives the difference between the connection and the standard metric-connection, while the torsion tensor $T_{mn}{}^p = \Gamma_{mn}{}^p - \Gamma_{nm}{}^p$ gives the antisymmetric component of the connection. The relation between the two is $\kappa_{mnp} = \frac{1}{2}(T_{mnp} - T_{npm} + T_{pmn})$.

As discussed in Refs. [24], the contorsion can be viewed as an $\mathfrak{so}(6)$ Lie algebra valued 1-form, and decomposes as $\kappa = \kappa^{\text{su}(3)} + \kappa^0$, where κ^0 lies in $\mathfrak{so}(6)/\mathfrak{su}(3)$. The action of $\kappa^{\text{su}(3)}$ on χ vanishes, since χ is an $\text{SU}(3)$ singlet. So, it is actually only the intrinsic contorsion κ_0 that contributes to Eq. (10).

When the torsion vanishes, there is no difference between the exterior derivative and a “covariant exterior derivative” obtained by replacing partial derivatives with covariant derivatives. However, when the torsion is nonzero and one makes the same substitution, only the symmetric part of the connection drops out, and the antisymmetric torsion part remains. The result is a twisted exterior derivative $d^{(T)}$ defined by

$$d^{(T)} \omega = d\omega - T^p \wedge \iota_p \omega, \quad \text{where} \quad T^p = T_{mn}{}^p dx^m \wedge dx^n. \quad (11)$$

⁴See Ref. [30] Sec. 7.2.6 for a pedagogical discussion of torsion and contorsion, and Ref. [25] Sec. 2.6 for a discussion of intrinsic torsion.

Just as the deformation space of a Calabi-Yau metric is determined by the d -cohomology ring $H^*(\mathcal{X}, \mathbb{C})$, one expects the natural deformation space of an $SU(3)$ structure metric to be closely related to the $d^{(T)}$ -cohomology ring $H_{(T)}^*(\mathcal{X}, \mathbb{C})$. An analogous expectation holds in context of $SU(2)$ structures; we will see how this expectation is realized in Sec. 6.

2.2 $SU(2)$ structure in 6D

2.2.1 Spinors and $SU(2)$ invariant tensors

When there exist not one, but two global nowhere-vanishing everywhere-independent positive chirality spinors on \mathcal{X} , the structure group of the frame bundle is further reduced to $SU(2)$. We assume without loss of generality that the two spinors are normalized, $\chi_1^\dagger \chi_1 = \chi_2^\dagger \chi_2$, and orthogonal, $\chi_1^\dagger \chi_2 = 0$. The spinors determine a triple of tensors $(J^\alpha)_a{}^b$, $\alpha = 1, 2, 3$, and a pair of real 1-forms w^1, w^2 on \mathcal{X} ,

$$(J^\alpha)_a{}^b = \frac{i}{2} \chi^\dagger \sigma^\alpha \gamma_a{}^b \chi, \quad w_a^\alpha = \frac{1}{2} \chi^T \sigma^\alpha \gamma_a \chi, \quad \text{where} \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \quad (12)$$

Here, σ^α are the 2×2 Pauli matrices, satisfying $\sigma^\alpha \sigma^\beta = \delta^{\alpha\beta} + i\epsilon^{\alpha\beta\gamma} \sigma^\gamma$. The 1-form w^3 vanishes. The condition that w^1, w^2 be real is a relative phase convention on χ_1 and χ_2 . An overall phase rotation of χ acts as $SO(2)$ rotation on w^1 and w^2 . Lowering the second index of $(J^\alpha)_a{}^b$, we obtain a triple of 2-forms $J^\alpha = \frac{1}{2} (J^\alpha)_{ab} e^a \wedge e^b$, which can be written

$$J^1 = j^1, \quad J^2 = j^2, \quad J^3 = j^3 + w^1 \wedge w^2, \quad (13)$$

with the following interpretation.

2.2.2 Bundle decompositions and almost hypercomplex structure

The coframe bundle $F^*\mathcal{X}$ splits as the sum of a 4D subbundle $F_4^*\mathcal{X}$ of structure group $SU(2)$ and 2D trivial subbundle $F_2^*\mathcal{X}$. The $(j^\alpha)_\alpha{}^\beta$ give a triple of almost complex structures satisfying the quaternionic algebra, on $F_4^*\mathcal{X}$. The forms w^1 and w^2 trivialize $F_2^*\mathcal{X}$. On $F_2^*\mathcal{X}$, there is a single almost complex structure, with holomorphic 1-form $v_1 + iv_2$.

Since the frame bundle splits, the spinor bundle correspondingly factorizes as a $\text{Spin}(4)$ subbundle times $\text{Spin}(2)$ subbundle. The $\text{Spin}(4)$ subbundle, of structure group $SU(2)$, admits global nowhere-vanishing spinors $\chi_\pm^{(4)}$, where the subscripts give the $\gamma_{(4)} = \gamma_{1234}$ chirality. The $\text{Spin}(2)$ bundle, of trivial structure group, admits global nowhere-vanishing spinors $\chi_\pm^{(2)}$, where the subscripts give the $\gamma_{(2)} = i\gamma_{56}$ chirality. Here, components of gamma matrices are with respect to a frame basis that respects the split $F\mathcal{X} = F_4\mathcal{X} + F_2\mathcal{X}$. The pair of global nowhere-vanishing positive chirality spinors is $\chi_1 = \xi_+ \otimes \zeta_+$ and $\chi_2 = \xi_- \otimes \zeta_-$.

Introducing 4D and 2D gamma matrices, $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}_\rho$, respectively,

$$\{\tilde{\gamma}_\alpha, \tilde{\gamma}_\beta\} = 2\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, 4, \quad \text{and} \quad \{\tilde{\gamma}_\rho, \tilde{\gamma}_\sigma\} = 2\delta_{\rho\sigma}, \quad \rho, \sigma = 5, 6, \quad (14)$$

we can write the 6D gamma matrices as $\gamma_\rho = 1 \otimes \tilde{\gamma}_\rho$ and $\gamma_\alpha = \gamma_{(2)}(\tilde{\gamma}_\alpha \otimes 1)$. Here, $\gamma_{(4)} = \tilde{\gamma}_{(4)} \otimes 1$, $\gamma_{(2)} = 1 \otimes \tilde{\gamma}_{(2)}$, and $\gamma_{(6)} = \tilde{\gamma}_{(4)} \otimes \tilde{\gamma}_{(2)}$, where $\tilde{\gamma}_{(4)}\tilde{\gamma}_{1234}$ and $\tilde{\gamma}_{(2)} = i\tilde{\gamma}_{56}$. The previous definitions are then equivalent to

$$(j^\alpha)_\alpha{}^\beta = \frac{i}{2}\xi^\dagger \sigma^\alpha \tilde{\gamma}_\alpha{}^\beta \xi \quad \text{and} \quad (w^1 + iw^2)_\rho = \zeta_+^T \tilde{\gamma}_\rho \zeta_-, \quad \text{where} \quad \xi = \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}. \quad (15)$$

This makes it clear that the 1-forms w^1 and w^2 are sections of $F_2^*\mathcal{X}$ and the 2-forms j^α (with lowered indices) are sections of $\wedge^2 F_4^*\mathcal{X}$.

In the Majorana convention assumed in Sec. 2.1, γ_a and $\gamma_{(6)}$ and $\gamma_{(2)}$ are imaginary and Hermitian, while $\gamma_{(4)}$ is real and Hermitian. Thus, complex conjugation of a 6D spinor reverses its $\gamma_{(6)}$ and $\gamma_{(2)}$ chiralities, leaving its $\gamma_{(4)}$ chirality unchanged. Similarly, $\tilde{\gamma}_\alpha$ is real and Hermitian, while $\tilde{\gamma}_\rho$ is imaginary and Hermitian. The simplest compatible phase convention for ξ_\pm and ζ_\pm is $\xi_\pm^* = \xi_\pm$ and $\zeta_\pm^* = \zeta_\mp$. This leads to a variety of alternative expressions for j^α and $w^1 + iw^2$. For example, of relevance below, we note that

$$(j^3)_\alpha{}^\beta = i\xi_+^\dagger \tilde{\gamma}_\alpha{}^\beta \xi_+, \quad \text{and} \quad (J^3)_\alpha{}^\beta = i\chi_1^\dagger \gamma_\alpha{}^\beta \chi_1. \quad (16)$$

From Fierz identities, the almost complex structures $(j^\alpha)_\alpha{}^\beta$ on $F_4^*\mathcal{X}$ satisfy

$$(j^\alpha)_\alpha{}^\gamma (j^\beta)_\gamma{}^\beta = -\delta^{AB} - \epsilon^{ABC} (j^C)_\alpha{}^\beta. \quad (17)$$

This is the quaternionic algebra up to an expected minus sign on the second term.⁵ We also have

$$J^A \wedge J^B = \delta^{AB} \text{Vol}_4, \quad w^1 \wedge w^2 = \text{Vol}_2, \quad \text{and} \quad \text{Vol}_6 = \text{Vol}_4 \wedge \text{Vol}_2, \quad (18)$$

where Vol_4 and Vol_2 are the volume forms associated to $F_2^*\mathcal{X}$ and $F_4^*\mathcal{X}$, respectively.

2.2.3 Torsion, contorsion, and twisted exterior derivative

When χ_1 and χ_2 are covariantly constant with respect to the standard spin connection ∇ , it is straightforward to show that dJ and $d(w^1 + iw^2)$ vanish. Then, the manifold \mathcal{X} has $\text{SU}(2)$ holonomy, and globally factorizes as $\text{K3} \times T^2$. More generally, as in the previous section, the spinors are instead covariantly constant with respect to a connection $\nabla^{(T)}$ with torsion,

$$\nabla_m^{(T)} \chi_{1,2} = (\nabla_m + \frac{1}{4} \kappa_{mnp} \gamma^{np}) \chi_{1,2} = 0. \quad (19)$$

As in the discussion of the previous section, we then expect a relation between the twisted cohomology defined by $d^{(T)}$ (cf. Eq. (11)) and the deformation space of $\text{SU}(2)$ structure metrics.

⁵the transpose of $(j^A)_\alpha{}^\beta$ acts on the frame rather than coframe bundle (i.e., tangent rather than cotangent bundle) and satisfies the quaternionic algebra with the standard sign conventions.

2.2.4 SU(3) structure from SU(2) structure

Finally, since $SU(2) \subset SU(3)$, we should be able to reproduce the results of Sec. 2.1 as a special case of this section, simply forgetting about χ_2 . This is indeed the case, and we find

$$J = j^3 + w^1 \wedge w^2, \quad \Omega = (j^1 + ij^2) \wedge (w^1 + iw^2). \quad (20)$$

The first relation follows from Eqs. (5) and (16), and the second follows from $\gamma_{\alpha\beta\rho} = \tilde{\gamma}_{ab} \otimes \tilde{\gamma}_\rho$.

2.3 Manifolds of SU(2) structure and SU(3) holonomy

There is an interesting intermediate case, in which the contorsion annihilates one of the two spinors of Sec. 2.2, say χ_1 , but not the other, so that

$$\nabla^{(T)}\chi_1 = \nabla^{(T)}\chi_2 = 0, \quad \text{and} \quad \nabla\chi_1 = 0, \quad \nabla\chi_2 \neq 0. \quad (21)$$

In this case, the same manifold \mathcal{X} is a Calabi-Yau manifold of SU(3) holonomy (with spinor χ_1), and also a manifold of SU(2) structure (with spinors χ_1, χ_2). The moduli space of Calabi-Yau metrics is a proper subspace of the moduli space of SU(2) structure metrics, and the ordinary de Rham cohomology ring $H^*(\mathcal{X}, \mathbb{R}) \cong \text{Harm}(\mathcal{X})$ is a proper subring of the twisted de Rham cohomology ring $H_{(T)}^*(\mathcal{X}, \mathbb{R}) \cong \text{Harm}_{(T)}(\mathcal{X})$.

In this paper, we analyze a class of Calabi-Yau manifolds for which a stronger statement holds, in *integer* cohomology. The Calabi-Yau manifolds are abelian fibrations over base $\mathcal{B} = \mathbb{P}^1$, and the torsion can be interpreted as a map $T: H_{(T)}^p(\mathcal{B}, R^q\pi_*\mathbb{Z}) \rightarrow H_{(T)}^{p+1}(\mathcal{B}, R^q\pi_*\mathbb{Z})$.⁶ We obtain the cohomology ring $H^*(\mathcal{X}, \mathbb{Z})$ by computing the d -cohomology on a space of representatives of $H_{(T)}^*(\mathcal{X}, \mathbb{Z})$. Thus, in a spectral-sequence-like sense, $H_{(T)}^*(\mathcal{X}, \mathbb{Z})$ gives a penultimate approximation to $H^*(\mathcal{X}, \mathbb{Z})$. This is similar to the logic of Ref. [38]. There, the twisted homology ring of an SU(3) structure manifold appeared as an approximation to its ordinary cohomology ring, within a spectral sequence.⁷ Here, the twisted cohomology of an SU(2) structure manifold appears as an approximation to its ordinary cohomology ring.

3 A class of abelian fibered Calabi-Yau manifolds $\mathcal{X}_{m,n}$

We focus on the class of Calabi-Yau 3-folds $\mathcal{X}_{m,n}$ that were studied in Ref. [33] (via duality) and Ref. [12] (intrinsically). The properties that will be relevant for us here are:

1. $\mathcal{X}_{m,n}$ is an abelian surface (T^4) fibration over \mathbb{P}^1 , with $8 + M$ singular fibers, $M \geq 0$.
2. The Hodge numbers of $\mathcal{X}_{m,n}$ are $h^{11} = h^{21} = M + 2$, where the M and the positive integers m, n , are constrained by $M + 4mn = 16$.

⁶Here, $H_{(T)}^p(\mathcal{B}, R^q\pi_*\mathbb{Z})$ is roughly the cohomology subgroup of degree p on the base and degree q on fiber.

⁷It so happens that the twisted cohomology ring in the example of Ref. [38] coincides with the ordinary cohomology ring of the mirror of the quintic Calabi-Yau 3-fold.

3. The polarization of the abelian fiber is $(\bar{m}, \bar{n}) = (m, n) / \gcd(m, n)$. This means that the Kähler form on the abelian fiber is proportional to $\bar{m}dy^1 \wedge dy^2 + \bar{n}dx^3 + dx^4$, a positive integer form that can be used to define a projective embedding.
4. The generic Mordell-Weil lattice of sections (mod torsion) of $\mathcal{X}_{m,n}$ is D_M . Here, generic means that all singular fibers are topologically I_1 times an elliptic curve (T^2).
5. The generic Mordell-Weil subgroup of torsion sections is $\text{MW}_{\text{tor}} = \mathbb{Z}_m \times \mathbb{Z}_n$.
6. The fundamental group is $\pi_1 = \mathbb{Z}_n \times \mathbb{Z}_n$.
7. In a convenient basis, the nonzero intersection numbers are

$$H^2 \cdot A = 2\bar{m}\bar{n}, \quad H \cdot \mathcal{E}_I \cdot \mathcal{E}_J = -\bar{m}\delta_{IJ}, \quad (22)$$

where A is the class of the abelian fiber. This basis is not quite integral. Rather, H and \mathcal{E}_I are moduli dependent. However, for the appropriate choice of moduli, the forms $H - \frac{\bar{m}^2\bar{n}}{6}A$ and $\bar{m}\bar{n}\mathcal{E}_I$ give half-integer classes in $H(\mathcal{X}_{m,n}, \frac{1}{2}\mathbb{Z})$, while A , $H - \frac{\bar{m}^2\bar{n}}{6}A \pm \mathcal{E}_I$ and $\bar{m}\bar{n}(\mathcal{E}_I - \mathcal{E}_J)$ give integer classes in $H(\mathcal{X}_{m,n}, \mathbb{Z})$. This is discussed in Sec. 6.4.3.

8. The second Chern class of $\mathcal{X}_{m,n}$ is the sum of the singular loci of singular fibers ($8 + M$ elliptic curves). Its only nonzero intersection with H, A, \mathcal{E}_I is

$$H \cdot c_2 = 8 + M. \quad (23)$$

A few additional observations will be useful. Since π_1 is abelian, $H_1(\mathcal{X}_{m,n}, \mathbb{Z}) = \pi_1(\mathcal{X}_{m,n}) = \mathbb{Z}_n \times \mathbb{Z}_n$, which can be shown to be generated by the classes of two S^1 's in the generic fiber. In addition to the group $\text{MW}_{\text{tor}} = \mathbb{Z}_m \times \mathbb{Z}_n$ of torsion sections, the T^2 product of the two torsion S^1 's is a $\mathbb{Z}_{\gcd(m,n)}$ torsion class, where $\gcd(m, n)$ is the greatest common divisor of m and n . (It is completely vertical, so it is not a section in MW_{tor} .) This exhausts all torsion 2-cycles. Therefore, the complete integer homology ring is

$$\begin{aligned} H_0(\mathcal{X}_{m,n}, \mathbb{Z}) &= \mathbb{Z}, \\ H_1(\mathcal{X}_{m,n}, \mathbb{Z}) &= \mathbb{Z}_n \times \mathbb{Z}_n, \\ H_2(\mathcal{X}_{m,n}, \mathbb{Z}) &= \mathbb{Z}^{M+2} \times \mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_{\gcd(m,n)}, \\ H_3(\mathcal{X}_{m,n}, \mathbb{Z}) &= \mathbb{Z}^{2(M+3)} \times \mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_{\gcd(m,n)}, \\ H_4(\mathcal{X}_{m,n}, \mathbb{Z}) &= \mathbb{Z}^{M+2} \times \mathbb{Z}_n \times \mathbb{Z}_n, \\ H_5(\mathcal{X}_{m,n}, \mathbb{Z}) &= \emptyset, \\ H_6(\mathcal{X}_{m,n}, \mathbb{Z}) &= \mathbb{Z}. \end{aligned} \quad (24)$$

To arrive at this list, the free part follows from the Hodge numbers, and the torsion part is obtained from $H_1^{\text{tor}} = \mathbb{Z}_n \times \mathbb{Z}_n$ and $H_2^{\text{tor}} = \mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_{\gcd(m,n)}$ as follows. Poincaré duality

implies that $H_p^{\text{tor}} = H_{\text{tor}}^{6-p}$, and the Universal Coefficient Theorem implies that $H_p^{\text{tor}} = H_{\text{tor}}^{p+1}$. Thus, on general grounds, for a connected 6D orientable manifold,

$$\begin{aligned} H_1^{\text{tor}} &= H_4^{\text{tor}} & (= H_{\text{tor}}^2 = H_{\text{tor}}^5), \\ H_2^{\text{tor}} &= H_3^{\text{tor}} & (= H_{\text{tor}}^3 = H_{\text{tor}}^4), \end{aligned} \quad (25)$$

along with $H_0^{\text{tor}} = H_5^{\text{tor}} = H_6^{\text{tor}} = 0$ ($= H_{\text{tor}}^1 = H_{\text{tor}}^6 = H_{\text{tor}}^0$). For $\mathcal{X}_{m,n}$, these torsion groups become

$$\begin{aligned} H_1^{\text{tor}} &= H_4^{\text{tor}} & (= H_{\text{tor}}^2 = H_{\text{tor}}^5) &= \mathbb{Z}_n \times \mathbb{Z}_n, \\ H_2^{\text{tor}} &= H_3^{\text{tor}} & (= H_{\text{tor}}^3 = H_{\text{tor}}^4) &= \mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_{\text{gcd}(m,n)}, \end{aligned} \quad (26)$$

3.1 Intrinsic definitions and properties

Two constructions of $\mathcal{X}_{m,n}$ were given in Ref. [12]. The first construction presented the explicit monodromy matrices of the $M + 8$ singular fibers, and described the homology via the algebra of (p, q, r, s) string junctions on the \mathbb{P}^1 base. The second construction, for the special case $m = n = 1$, realized $\mathcal{X}_{1,1}$ as the relative Jacobian of a genus-2 fibration over \mathbb{P}^1 . We now summarize this construction, as it will provide us with a useful intuition for the general case.

First, recall that an abelian variety is projective variety that is also an abelian group. It is a complex torus $T^{2d} \cong \mathbb{C}^d / \Lambda$, for some lattice Λ , with group addition and additive identity inherited from \mathbb{C}^2 . The fact that it is a projective variety implies that the Kähler form is proportional to a Hodge form [22]

$$dx^1 \wedge dx^2 + \delta_1 dx^3 \wedge dx^4 + \dots \delta_g dx^{2g-1} \wedge dx^{2g}, \quad (27)$$

with δ_i divisible by δ_{i-1} . The Poincaré dual homology class is represented by a Hodge divisor. The case $g = 1$ gives an elliptic curve and the case $g = 2$ gives an abelian surface. The g -tuple $(1, \delta_1, \dots, \delta_g)$ is known as the polarization of the abelian variety.

The Jacobian of a genus- g curve C_g is the space of degree zero line bundles on C_g , modulo linear equivalence. It is an abelian variety, which can be seen as follows. $\text{Jac}(C_g)$ can be thought of as the space of Wilson lines, i.e., a $U(1)$ phase for each of the independent 1-cycles generating $H_1(C_g, \mathbb{Z}) = \mathbb{Z}^{2g}$. Thus $\text{Jac}(C_g) \cong T^{2g}$. For a genus-2 curves, the Jacobian is an abelian surface (T^4). In general, the space of degree d line bundles over a space \mathcal{X} is referred to as the Picard group $\text{Pic}^d(\mathcal{X})$. Thus $\text{Jac}(C_g) = \text{Pic}^0(C_g)$. Jacobian varieties are always principally polarized, meaning $\delta_i = 1$ for $i = 1, \dots, g$.

Given a surface S that is fibered by genus- g curves over a base curve \mathcal{B} , the relative Jacobian $\text{Pic}^0(S/\mathcal{B})$, is obtained from S by replacing each fiber by its Jacobian. This gives an abelian fibration over \mathcal{B} . For $g = 2$ and $\mathcal{B} = \mathbb{P}^1$, we obtain an abelian surface fibration over \mathbb{P}^1 . For the appropriate choice of S , it was shown in Ref. [12], that $\mathcal{X}_{1,1} = \text{Pic}^0(S/\mathcal{B})$.

For $\mathcal{X}_{1,1}$, we have $M = 12$, and the homology group $H_4(\mathcal{X}_{1,1}, \mathbb{Z})$ is generated by the abelian fiber A together with the $2M$ theta divisors Θ_I, Θ'_I , for $I = 1, \dots, M$. (See the discussion in App. B.) It can be shown that $[\Theta_I + \Theta'_I] = [D]$, independent of I , so this

indeed gives $\dim(H^4) = M + 2$. The relation to the basis with convenient intersections (22) is $[\mathcal{E}_I] = \frac{1}{2}[\Theta_I - \Theta'_I]$ and $[H] = \frac{1}{2}[D] - \frac{1}{6}[A]$, where $[D] = [\mathcal{E}_J + \mathcal{E}'_J]$.

The generalizations of these statements for the Calabi-Yau manifold $\mathcal{X}_{m,n}$ can be found in Sec. 6.4 and App. B.

3.2 Duality to type IIB orientifolds with flux

3.2.1 The duality map leading to $\mathcal{X}_{m,n}$

The family of Calabi-Yau manifolds $\mathcal{X}_{m,n}$ was first studied in Ref. [33] in the context of a duality map relating the simplest type IIB flux compactifications—toroidal orientifold with flux—to purely geometric Calabi-Yau compactifications of type IIA string theory. As discussed in Ref. [33], the type IIB T^6/\mathbb{Z}_2 orientifold with the choice of $\mathcal{N} = 2$ flux [26]

$$\begin{aligned} F_{(3)}/((2\pi)^2\alpha') &= 2m(dx^4 \wedge dx^6 + dx^5 \wedge dx^7) \wedge dx^9, \\ H_{(3)}/((2\pi)^2\alpha') &= 2n(dx^4 \wedge dx^6 + dx^5 \wedge dx^7) \wedge dx^8, \end{aligned} \tag{28}$$

is dual to type IIA compactified on $\mathcal{X}_{m,n}$. The duality map has two steps: (i) T-duality of $T^3(x^4x^5x^9)$, which give a type IIA orientifold with M D6-branes and eight O6 planes, followed by (ii) an M-theory “9,10-flip.” The 9,10-flip lifts the type IIA background to M-theory to give a new x^{10} circle, and then compactifies on x^9 to return to type IIA string theory in 10 dimensions. The M-theory geometry resulting from the lift of the IIA orientifold is $\mathbb{R}^{1,3}(x^0x^1x^2x^3) \times \mathcal{X}_{m,n}(x^4x^5x^6x^7x^8x^{10}) \times S^1(x^9)$, where $\mathcal{X}_{m,n}$ is an abelian surfaces fibration over $\mathbb{P}^1(x^6x^7)$, i.e., a holomorphic T^4 fibration with additional structure (a zero section, addition of sections, and a theta divisor). The $S^1(x^{10})$ is contained in $T^4(x^4x^5x^8x^{10})$. From here, we compactify on $S^1(x^9)$ to obtain type IIA string theory on $\mathbb{R}^{1,3}(x^0x^1x^2x^3) \times \mathcal{X}_{m,n}(x^4x^5x^6x^7x^8x^{10})$.

3.2.2 Analogy to $K3 \times T^2$

The 3-fold $\mathcal{X}_{m,n}$ can be thought of as a twisted analog of $K3 \times T^2$, with fewer than 16 exceptional divisors. Recall that an elliptic K3 is an elliptic fibration over \mathbb{P}^1 , generically with 24 I_1 Kodaira fibers. Thus $K3 \times T^2$ is trivially a T^4 fibration over \mathbb{P}^1 , with a $T^2 \subset T^4$ factorizing. In $\mathcal{X}_{m,n}$, the T^4 no longer factorizes, and the number of singular fibers is reduced from 24 to $M + 8$, where $M = 16 - 4mn$. In the T^6/\mathbb{Z}_2 dual, M is the number of D3-branes. The Hodge numbers of $\mathcal{X}_{m,n}$ are $h^{11} = h^{21} = M + 2$, with possible values $M + 2 = 2, 6, 10, 14$.

3.2.3 Supersymmetry, spinors, and $SU(2)$ structure

Just as choice of flux in the dual type IIB T^6/\mathbb{Z}_2 orientifold spontaneously breaks $\mathcal{N} = 4$ to $\mathcal{N} = 2$ supersymmetry, the *topology* of $\mathcal{X}_{m,n}$ spontaneously breaks $\mathcal{N} = 4$ to $\mathcal{N} = 2$ supersymmetry in the dual type IIA compactification on $\mathcal{X}_{m,n}$. When the \mathbb{P}^1 base of $\mathcal{X}_{m,n}$ is large compared to the T^4 fiber, there is a hierarchy of scales, and it is meaningful to describe this supersymmetry breaking in the low energy effective field theory. In the absence of flux

in IIB, the $\mathcal{N} = 4$ supersymmetry is unbroken, and the dual IIA compactification manifold is $K3 \times T^2$.

The doubled ($\mathcal{N} = 4$) supersymmetry of $K3 \times T^2$ relative to the $\mathcal{N} = 2$ of a Calabi-Yau 3-fold follows from the fact that the latter admits one covariantly constant positive chirality spinor, whereas the former admits two: $u_+^{K3} \otimes u_+^{T^2}$ and $u_-^{K3} \otimes u_-^{T^2}$. Here, u_\pm^X is a \pm chirality spinor on the space X . Orientation reversal on X reverses the chirality of X . For $X = T^2$, this can be implemented by complex conjugation, and for $X =$ an elliptic K3, it can be implemented by complex conjugation of the elliptic fiber. Thus, viewing $K3 \times T^2$ as a T^4 fibration over \mathbb{P}^1 , the two positive chirality spinors are related by complex conjugation of the T^4 fiber.

The last paragraph carries over to $\mathcal{X}_{m,n}$, with one modification. Complex conjugation of the T^4 fiber of $\mathcal{X}_{m,n}$ indeed relates the standard covariantly constant positive chirality spinor to another global positive chirality spinor. However, complex conjugation of the fiber alone “breaks” the complex structure. The complex conjugate T^4 fibration is no longer a holomorphic T^4 fibration over \mathbb{P}^1 , due to the m, n dependent monodromies about singular fibers. Correspondingly, the new positive chirality spinor is not covariantly constant with respect to the standard spin connection. It together with its negative chirality counterpart generates a spontaneously broken $\mathcal{N} = 2$ supersymmetry in $\mathcal{N} = 4$. On the other hand, this new spinor *is* covariantly constant relative to a connection with torsion. The pair of positive chirality spinors (together with their negative chirality counterparts) gives rise to an $SU(2)$ structure on $\mathcal{X}_{m,n}$.

4 First-order $SU(2)$ structure metric and moduli space

The type IIB T^6/\mathbb{Z}_2 orientifold with flux [26, 32] gives an $\mathcal{N} = 4$ low energy supergravity theory valid below the compactification scale $1/R$. We assume this scale to be hierarchically smaller than string scale.⁸ The fluxes parametrize a gauging of this $\mathcal{N} = 4$ supergravity theory known as a flat gauging [1], which spontaneously breaks the $\mathcal{N} = 4$ to $\mathcal{N} = 2$ at an energy scale α'/R^3 . Intuitively, this energy scale arises since the periods of the fluxes over 3-cycles of volume $\sim R^3$ are quantized in units $\sim \alpha'$.

At energies $\alpha'/R^3 < E < 1/R$, the light scalars in the closed string sector are the 21 metric moduli of T^6 , 15 moduli from the RR 4-form axions $C_{(4)}$ with purely internal indices, and the axion-dilaton $C_{(0)} + ie^{-\phi}$. In addition, there are $6M$ scalars from the T^6 positions of M D3-branes (plus images). At energies below α'/R^3 , 10 metric moduli and 6 of the $C_{(4)}$ axions remain, together with all $6M$ D3 moduli.

When this is mapped to the type IIA compactification on $\mathcal{X}_{m,n}$, the same $\mathcal{N} = 4$ low energy supergravity theory is valid below the compactification scale $1/V_X^{1/6}$. The $\mathcal{N} = 4$ theory is spontaneously broken to $\mathcal{N} = 2$ at the scale $1/V_{\mathbb{P}^1}^{1/2}$ of the \mathbb{P}^1 base. The light scalars of the $\mathcal{N} = 4$ theory are $13 + 3M$ metric moduli $7 + M$ B -field moduli, and the dilaton. In the $\mathcal{N} = 2$ theory, the remaining light scalars are $6 + 3M$ metric moduli, $2 + M$ B -field moduli, and the dilaton.

⁸The tree level supergravity theory is no-scale model, in which the scale $1/R$ is not fixed.

4.1 Metric

Truncating to the tree level supergravity description of T^6/\mathbb{Z}_2 orientifold in type IIB, the chain of classical supergravity dualities discussed in Sec. 3.2 gives an approximate description of $\mathcal{X}_{m,n}$ based on a “first-order” $\mathcal{N} = 4$ metric of the form

$$ds_{\mathcal{N}=4}^2 = \sqrt{2V_4}(\Delta^{-1}ZG_{\alpha\beta}dx^\alpha dx^\beta + \Delta Z^{-1}(dx^4 + A)^2) + g_{\rho\sigma}\eta^\rho\eta^\sigma. \quad (29)$$

Here, the quantities η^ρ , for $\rho = 1, 2$, are global 1-forms on $\mathcal{X}_{m,n}$ defined by

$$\begin{aligned} \eta^1 &= dy^1 + A^1, \quad \text{where} \quad F^1 = dA^1 = 2ndx^1 \wedge dx^3, \\ \eta^2 &= dy^2 + A^2, \quad \text{where} \quad F^2 = dA^2 = 2ndx^2 \wedge dx^3. \end{aligned} \quad (30)$$

The coordinates $x^1, x^2, x^3, x^4, y^1, y^2$ have periodicity 1 and x coordinates are further identified under the involution

$$\mathcal{I}_4: (x^\alpha, x^4) \mapsto (-x^\alpha, -x^4). \quad (31)$$

This metric can be thought of as the twisted product of a 4D Gibbons-Hawking metric in the x^1, \dots, x^4 directions (quotiented by \mathbb{Z}_2) and a 2D torus metric in the y^1, y^2 directions. As we will see, the corresponding 4D + 2D split of the frame bundle of $\mathcal{X}_{m,n}$ realizes the $SU(2)$ structure.

For $m = n = 0$, this metric reduces to that of $K3 \times T^2$ in the approximation discussed in Ref. [35], whose notation we follow. For $m, n \neq 0$, the derivation is a minor generalization of that in Ref. [34], obtained by including all of the light $\mathcal{N} = 4$ degrees of freedom in the dual toroidal orientifold, and not just the exact moduli of the $\mathcal{N} = 2$ theory.

We now define the remaining quantities appearing in Eq. (29). The 2D metric $g_{\rho\sigma}$, for $\rho, \sigma = 1, 2$ is an arbitrary T^2 metric. The 3D metric $G_{\alpha\beta}$, for $\alpha, \beta = 1, 2, 3$, is an arbitrary T^3 metric, and $\Delta = \det^{1/2}(G)$. The quantity Z satisfies a Poisson equation on this T^3 ,

$$-\nabla_G^2 Z = \sum_{\text{sources } s} Q_s(\delta^3(\mathbf{x} - \mathbf{x}^s) - 1), \quad (32)$$

where s runs over $I = 1, \dots, M$, $I' = 1', \dots, M'$, and O_i for $i = 1, \dots, 8$. Here, (i) $Q_I = 1$ for a source at \mathbf{x}^I ; (ii) $Q_{I'} = 1$ for an image source at $\mathbf{x}^{I'} = -\mathbf{x}^I$; and (iii) $Q_{O_i} = -4$ for sources at the $2^3 = 8$ \mathbb{Z}_2 fixed points on T^3 where each of the x^α equals 0 or $\frac{1}{2}$. We adopt the convention

$$\int d^3x Z = 1. \quad (33)$$

A different convention can be absorbed into a redefinition of V_4 and $G_{\alpha\beta}$. The solution can be expressed as

$$Z = 1 + \sum_{\text{sources } s} Z_s, \quad \text{with} \quad Z_s(\mathbf{x}) = Q_s K(\mathbf{x}, \mathbf{x}^s), \quad (34)$$

where $K(\mathbf{x}, \mathbf{x}')$ is a Green’s function on T^3 satisfying

$$\nabla_G^2 K(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}') - 1, \quad \text{with} \quad \int d^3y K(\mathbf{x}, \mathbf{x}') = 0. \quad (35)$$

The connection A satisfies

$$dA = \star_G dZ - 2m(\eta^1 \wedge dx^2 - \eta^2 \wedge dx^1), \quad (36)$$

where \star_G denotes the Hodge star operator in the metric $G_{\alpha\beta}$ on T^3 . In what follows, it will be convenient to write

$$\eta^4 = dx^4 + A, \quad (37)$$

and to define A_I and $A_{I'}$ satisfying

$$dA_I = \star_G dZ_I, \quad dA_{I'} = \star_G dZ_{I'}. \quad (38)$$

4.2 Frame

Let $E^\alpha{}_\beta$ be a vielbein for $G_{\alpha\beta}$ and $e^\rho{}_\sigma$ be a vielbein for $g_{\rho\sigma}$,

$$G_{\alpha\beta} = \delta_{\gamma\delta} E^\gamma{}_\alpha E^\delta{}_\beta, \quad g_{\rho\sigma} = \delta_{\tau\nu} e^\tau{}_\rho e^\nu{}_\sigma. \quad (39)$$

The natural coframe for the metric (29) realizes a 6D \rightarrow 4D + 2D splitting of the coframe bundle $\mathcal{F}^* \mathcal{X}_{m,n}$ into two subbundles $\mathcal{F}_4^* \mathcal{X}_{m,n}$ and $\mathcal{F}_2^* \mathcal{X}_{m,n}$:

$$\begin{aligned} \text{4D coframe: } \theta^\alpha &= (2V_4)^{1/4} \Delta^{-1/2} Z^{1/2} E^{-1}{}^\alpha{}_\beta dx^\beta, \quad \theta^4 = (2V_4)^{1/4} \Delta^{1/2} Z^{-1/2} \eta^4. \\ \text{2D coframe: } w^\rho &= e^\rho{}_\sigma \eta^\sigma. \end{aligned} \quad (40)$$

The structure group of the 4D subbundle $\mathcal{F}_4^* \mathcal{X}_{m,n}$ spanned by the first line is $SU(2)$, and the quantities on the second line are global 1-forms trivializing the 2D subbundle $\mathcal{F}_2^* \mathcal{X}_{m,n}$.

4.3 Spin connection, torsion, and twisted exterior derivative

The decomposition $\mathcal{F}^* = \mathcal{F}_4^* \oplus \mathcal{F}_2^*$ means that the connection 1-form $\omega^A{}_B$ should decompose into an $\mathfrak{so}(4)$ -valued connections 1-form $\omega^m{}_n$ and $\mathfrak{so}(2)$ -valued connection 1-form $\omega^\rho{}_\sigma$. The statement that \mathcal{F}_4^* has structure group $SU(2)$ and \mathcal{F}_2^* has trivial structure group means that we should have $\omega^m{}_n \subset \mathfrak{su}(2) \subset \mathfrak{so}(4)$ and $\omega^\rho{}_\sigma = 0$, for the appropriate choice of torsion. Therefore, the first of the Cartan structure equations

$$d\theta^A + \omega^A{}_B \theta^B = T^A, \quad (41)$$

should take the form

$$d\theta^\alpha + \omega^\alpha{}_\beta \theta^\beta + \omega^\alpha{}_4 \wedge \theta^4 = T^\alpha, \quad d\theta^4 + \omega^4{}_\beta \theta^\beta = T^4, \quad dw^\rho = T^\rho, \quad (42)$$

where $T^A = (T^\alpha, T^4; T^\rho)$ is the torsion 2-form. We can alternatively write these equations as

$$d_T \theta^A + \omega^A{}_B \theta^B = 0, \quad (43)$$

where

$$d_T \theta^A = d\theta^A - T^A, \quad (44)$$

is the twisted exterior derivative (11) acting on 1-forms.

Let us define a twisted exterior derivative on $\mathcal{X}_{m,n}$ by

$$\begin{aligned} d_T(dx^\alpha) &= d(dx^\alpha), \quad \text{for } \alpha = 1, 2, 3, \\ d_T\eta^4 &= d\eta^4 + 2m(\eta^1 \wedge dx^2 - \eta^2 \wedge dx^1), \\ d_T\eta^1 &= d\eta^1 - 2ndx^1 \wedge dx^3, \\ d_T\eta^2 &= d\eta^2 - 2ndx^2 \wedge dx^3. \end{aligned} \tag{45}$$

Then, if we express the torsion 2-form in terms of the basis $(dx^\alpha, \eta^4; \eta^\rho)$ (and its dual) rather than $(\theta^\alpha, \theta^4; \theta^r)$ (and its dual), we have

$$\begin{aligned} T^\alpha &= 0, \\ T^4 &= -2m(\eta^1 \wedge dx^2 - \eta^2 \wedge dx^1), \\ T^1 &= 2ndx^1 \wedge dx^3, \\ T^2 &= 2ndx^2 \wedge dx^3. \end{aligned} \quad (\text{superscripts in the } (dx^\alpha, \eta^4; \eta^\rho) \text{ basis}) \tag{46}$$

In Sec. 5.1, we will interpret the topology of $\mathcal{X}_{m,n}$ as that of a T^4 fibration in the y^1, y^2, x^3, x^4 directions over a $\mathbb{P}^1 \cong T^2/\mathbb{Z}_2$ in the x^1x^2 directions. The fibration gives a splitting of the cohomology of $\mathcal{X}_{m,n}$,⁹ which decomposes the cohomology group $H_{p+q}(\mathcal{X}_{m,n}, \mathbb{Z})$ into the sum of subgroups $H_{(T)}^p(\mathbb{P}^1, R^q\pi_*\mathbb{Z})$ of degree p on the \mathbb{P}^1 base and q on the abelian surface fiber. The torsion is a map on differential forms preserving the fiber degree and increasing the base degree by 1. It can be interpreted as a map $T: H_{(T)}^p(\mathbb{P}^1, R^q\pi_*\mathbb{Z}) \rightarrow H_{(T)}^{p+1}(\mathbb{P}^1, R^q\pi_*\mathbb{Z})$.

With these definitions, η^α and η^ρ are twisted closed, and η^4 satisfies

$$d_T\eta^4 = \star_G dZ. \tag{47}$$

Therefore,

$$d_T^2(dx^\alpha) = d_T^2\eta^r = 0 \quad \text{and} \quad d_T^2\eta^4 = -\nabla_G^2 Z dx^1 \wedge dx^2 \wedge dx^3 \neq 0. \tag{48}$$

The last equation appears at first glance to contradict the property $d_T^2 = 0$ needed to construct a twisted cohomology ring from d_T . However, this is not the case, since the 1-forms dx^α and η^4 are odd under the the involution (31). A basis of \mathbb{Z}_2 invariant products is

$$dx^\alpha \wedge \eta^4 \quad \text{and} \quad \frac{1}{2}\epsilon_{\alpha\beta\gamma} dx^\beta \wedge dx^\gamma, \tag{49}$$

which are indeed annihilated by d_T^2 .

With this choice of torsion 2-form, it is possible to solve the Cartan structure equations (42) for the spin connection. We find

$$\begin{aligned} \omega^{\hat{\alpha}}_{\hat{\beta}} &= \frac{1}{2} \left(\partial_{\hat{\beta}} \log Z \theta^{\hat{\alpha}} - \partial^{\hat{\alpha}} \log Z \theta_{\hat{\beta}} \right) + \frac{1}{2} F_{\hat{4}\hat{\beta}}^{\hat{\alpha}} \theta^{\hat{4}}, \\ \omega^{\hat{4}}_{\hat{\beta}} &= -\omega^{\hat{\beta}}_{\hat{4}} = -\frac{1}{2} \partial_{\hat{\beta}} \log Z \theta^{\hat{4}} - \frac{1}{2} F_{\hat{\alpha}\hat{\beta}}^{\hat{4}} \theta^{\hat{\alpha}}, \end{aligned} \tag{50}$$

⁹See the discussion of Leray spectral sequences in Ref. [22]. In general, the fibration gives a filtration of the cohomology, but when the fiber is Kähler, as is the case here, we obtain a splitting.

with all other components vanishing. Therefore, the spin connection indeed decomposes in the way described at the beginning of this section. Here, we have included hats to clearly distinguish frame indices from coordinate indices. We define $F^4 = \theta^4{}_4 F = (2V_4)^{1/4} \Delta^{1/2} Z^{-1/2} F$, where $F = dA$. The other quantities appearing in Eq. (50) are defined in the standard way.

4.4 Twisted harmonic forms

1-forms

The \mathbb{Z}_2 invariant 1-forms η^1 and η^2 are annihilated by d_T , as are their Hodge duals, which are proportional to $\eta^2 \wedge \gamma$ and $\eta^1 \wedge \gamma$,

$$\gamma = 2Z dx^1 \wedge dx^2 \wedge dx^3 \wedge \eta^4 = 2Z dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4. \quad (51)$$

Therefore, η^1 and η^2 are twisted harmonic 1-forms on $\mathcal{X}_{m,n}$. The 4-form γ is related to the volume form on \mathcal{F}_4^* by

$$\text{Vol}_{(4)} = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = V_4 \gamma. \quad (52)$$

2-forms

The 4D coframe $\mathcal{F}_4^* \mathcal{X}_{m,n}$ admits an almost hypercomplex structure $(\mathcal{J}^\alpha)_m{}^n$, for $\alpha = 1, 2, 3$, whose corresponding triple of 2-forms $(j^\alpha)_{mn} = (j^\alpha)_m{}^p g_{pn}$ is

$$\begin{aligned} j^1 &= \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3, \\ j^2 &= \theta^2 \wedge \theta^4 + \theta^3 \wedge \theta^1, \\ j^3 &= \theta^3 \wedge \theta^4 + \theta^1 \wedge \theta^2. \end{aligned} \quad (53)$$

These differential forms are closed with respect to the twisted exterior derivative d_T , and are selfdual with respect to the Hodge star operator \star_4 on \mathcal{F}_4^* , defined from the metric

$$ds_4^2 = (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 + (\theta^4)^2. \quad (54)$$

Since the 6D Hodge star operator acts on these forms as $\star = w^1 \wedge w^2 \wedge \star_4$, with w^1 and w^2 closed, we conclude that both j^α and $\star j^\alpha$ are closed. That is, the j^α are twisted harmonic forms on $\mathcal{X}_{m,n}$, annihilated by the twisted Laplace-de Rham operator

$$\Delta_T = d_T d_T^\dagger + d_T^\dagger d_T, \quad (55)$$

where $d_T^\dagger = \star d_T \star$. To obtain the complete list of twisted harmonic 2-forms, the steps are analogous to those for the harmonic forms of K3 in the corresponding approximate metric [35]. We follow Sec. 3.4 of Ref. [35] closely in the remainder of this section.

The j^α can be decomposed as sums of pairs of closed 2-forms

$$j^\alpha = \sqrt{\frac{V_4}{2}} (\omega^\alpha + \omega_\alpha), \quad (56)$$

where

$$\begin{aligned}\omega^\alpha &= 2\bar{\theta}^\alpha \wedge \bar{\theta}^4 + \left(\frac{Z-1}{Z}\right)\epsilon_{\alpha\beta\gamma}\bar{\theta}^\beta \wedge \bar{\theta}^\gamma + d_T\bar{\lambda}_\alpha, \\ \omega_\alpha &= \frac{1}{Z}\epsilon_{\alpha\beta\gamma}\bar{\theta}^\beta \wedge \bar{\theta}^\gamma - d_T\bar{\lambda}_\alpha.\end{aligned}\tag{57}$$

Here, we have included a possible twisted exact term in the definition of ω^α and ω_α , and bars denote a coframe with respect to a “unit” 4D metric $d\bar{s}_4^2$, defined by $\theta^m = (2V_4)^{1/2}\bar{\theta}^m$. We choose $\bar{\lambda}_\alpha$ so that ω^α and ω_α are twisted harmonic. Then $\bar{\lambda}_\alpha$ satisfies

$$\sqrt{\frac{V_4}{2}}(d_T\bar{\lambda}_\alpha + \star_4 d_T\bar{\lambda}_\alpha) = \frac{1-Z}{Z}j^\alpha.\tag{58}$$

Finally, the following anti-selfdual 2-forms can be shown to be twisted harmonic:

$$\begin{aligned}\omega_I &= \left(\frac{Z_I - Z_{I'}}{Z}\right)_{,\alpha} \left(dx^\alpha \wedge (dx^4 + A) - \frac{Z}{2}G^{\alpha\alpha'}\epsilon_{\alpha'\beta\gamma}dx^\beta \wedge dx^\gamma\right) \\ &= -d\left((A_I - A_{I'}) - \frac{(Z_I - Z_{I'})}{Z}(dx^4 + A)\right), \quad \text{for } I = 1, \dots, M.\end{aligned}\tag{59}$$

In summary, a basis of twisted harmonic 2-forms is

$$\omega_a = (\omega^\alpha, \omega_\alpha, \omega_I), \quad \text{together with } \eta^1 \wedge \eta^2.\tag{60}$$

It is possible to show by an analogous computation to that in App. E.4 of Ref. [35] that

$$\int_{\mathcal{X}_{m,n}} \eta^1 \wedge \eta^2 \wedge \omega_a \wedge \omega_b = \eta_{ab},\tag{61}$$

where η_{ab} takes the block form

$$\eta = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.\tag{62}$$

Note that $\omega^\alpha \wedge \omega_\beta = 2\gamma\delta^\alpha_\beta$ and that, from Eq. (61), $\omega_I \wedge \omega_J$ is in the same twisted cohomology class as $-\gamma\delta_{IJ}$.

Higher degree forms

The twisted harmonic forms of higher degree are obtained from products of the twisted harmonic 1-forms and 2-forms listed above. Thus, we see that $\text{Harm}(\mathcal{X}_{m,n}) = \text{Harm}_{(4)} \times \text{Harm}_{(2)}$, where

$$\text{Harm}_{(4)} = \langle 1, \omega_a, \gamma \rangle \quad \text{and} \quad \text{Harm}_{(2)} = \langle 1, \eta^\rho \rangle.\tag{63}$$

Here, angle brackets denote the span of the quantities enclosed. The cohomology ring similarly factorizes.

4.5 Moduli space of almost hypercomplex structure on \mathcal{F}_4^*

The metric (29) depends explicitly on moduli V_4 , $G_{\alpha\beta}$, $x^{I\alpha}$, $g_{\rho\sigma}$ and implicitly on three additional moduli $\beta^{\alpha\beta}$, defined as follows. Since Eq. (36) determines A only up to an additive shift by a constant 1-form $\beta_\alpha dx^\alpha$, we can write

$$A = A^0 + \beta_\alpha dx^\alpha, \quad (64)$$

where A^0 is a fiducial connection independent of E and β , but depending on M sources locations \mathbf{x}^I on T^3 . It is convenient to trade β_α for a bivector $\beta^{\alpha\beta} = \epsilon^{\alpha\beta\gamma} \beta_\gamma$.

Since the Hodge star operation depends on the metric moduli, so do the harmonic forms. It is straightforward to see that the harmonic forms of the previous section do not depend on $g_{\rho\sigma}$. The η^ρ are moduli independent, and the ω_a depend on E , β , and x . Let us define moduli independent forms ξ_a by

$$\xi_a = \omega_a \Big|_{(E, \beta, x) = (1, 0, 0)}. \quad (65)$$

Then, for suitable conventions on A^0 , one can show that the twisted cohomology classes of the ω_a and ξ_a are related by

$$[\omega_a] = V(E, \beta, x)_a{}^b [\xi_b], \quad (66)$$

where

$$\begin{aligned} V(E, \beta, x) &= V(E)V(\beta)V(x) = \begin{pmatrix} E & 0 & 0 \\ 0 & E^{-1T} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta & -0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x^T x & 2x^T \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} \\ &= \begin{pmatrix} E & -EC & 2Ex^T \\ 0 & E^{-1T} & 0 \\ 0 & x & 1 \end{pmatrix}, \end{aligned} \quad (67)$$

with $C^{\alpha\beta} = \beta^{\alpha\beta} - \delta_{IJ} x^{I\alpha} x^{J\beta}$. The nontrivial part of this relation to prove is the x dependence. Setting $E = 1$ and $\beta = 0$ for simplicity, the identification of the x dependence of both sides of Eq. (66) is closely analogous to that of the corresponding K3 discussion in Sec. 3.4 of Ref. [35], to which we refer the reader for further discussion. The $x^{I\alpha}$ correspond to D6 positions on T^3 in the D6/O6 dual of the type IIB T^6/\mathbb{Z}_2 orientifold after T-dualizing a T^3 in the T^6 . The choice $x^{I\alpha} = 0$ for $I = 1, \dots, M$ describes a metric (29) with a curve of D_M singularities, and corresponds to choosing all M D-branes coincident in the dual orientifold with $\mathcal{N} = 2$ flux, yielding enhanced $SO(2M)$ gauge symmetry.

Comparing Eqs. (56) and (66), we see that the matrix $V(E, \beta, x)$ parametrizes the moduli space of almost hypercomplex structure on \mathcal{F}_4^* . Choices of ω_α differing by $SO(3)$ rotation of $(\omega^\alpha, \omega_\alpha)$ and $SO(3 + M)$ rotation of ω_I gives the same almost hypercomplex structure (AHS).¹⁰ The moduli matrix $V(E, \beta, x)$ is best viewed as a vielbein for the coset

$$\mathcal{M}_{\text{AHS}} = (SO(3) \times SO(3 + M)) \backslash SO(3, 3 + M) / \Gamma_{3, 3 + M}, \quad (68)$$

¹⁰Note that ω^α and ω_α are in the same representation of $SO(3)$, since $O^{-1T} = O$ for $O \in SO(3)$.

where $\Gamma_{3,3+M}$ is the discrete group of lattice isomorphisms of the twisted cohomology lattice of $\mathcal{F}_4^* \mathcal{X}_{m,n}$. The $\text{SO}(3) \times \text{SO}(3+M)$ acts on the left (frame) index of V_a^b , and $\Gamma_{3,3+M}$ acts on the right index of V_a^b by Eq. (66) since it induces an action on ξ_b . The space of 4D metrics on \mathcal{F}_4^* is the space $\mathbb{R}_{>0}^1$ of overall volume modulus V_4 times the space of almost hypercomplex structure \mathcal{M}_{AHS} .

4.6 Moduli space of almost complex structure on \mathcal{F}_2^*

An almost complex structure can be defined on $\mathcal{F}_2^* \mathcal{X}_{m,n}$ by taking the forms of type (1,0) and (0,1) to be

$$w = w^1 + iw^2 \quad \text{and} \quad \bar{w} = w^1 - iw^2. \quad (69)$$

Parametrizing the 2D metric on \mathcal{F}_2^* as

$$ds_2^2 = g_{\rho\sigma} \eta^\rho \eta^\sigma = \frac{V_2}{\text{Im } \tau_2} |\eta^1 + \tau_2 \eta^2|^2, \quad (70)$$

this is equivalent to

$$w = \left(\frac{V_2}{\text{Im } \tau_2} \right)^{1/2} \eta, \quad \text{where} \quad \eta = \eta^1 + \tau_2 \eta^2. \quad (71)$$

The space of 2D metrics $g_{\alpha\beta}$ is

$$\text{SO}(2) \backslash \text{GL}(2) / \Gamma_2, \quad (72)$$

from the choice of vielbein $w^\rho_\sigma \in \text{GL}(2)$, modulo $\text{SO}(2)$ rotation on the left and change of twisted cohomology lattice basis $\Gamma_2 \cong \text{GL}(2, \mathbb{Z})$ on the right. Equivalently, it is the space $\mathbb{R}_{>0}^1$ of overall volume modulus V_2 times the space

$$\mathcal{M}_{\text{ACS}} = U(1) \backslash \text{SL}(2) / \text{PSL}(2, \mathbb{Z}) \quad (73)$$

of almost complex structure moduli τ_2 .

4.7 Moduli space of $\text{SU}(2)$ structure metrics on $\mathcal{X}_{m,n}$.

We have shown that the moduli V_4 , $G_{\alpha\beta}$, $x^{I\alpha}$ and $g_{\alpha\beta}$ of the first-order $\mathcal{N} = 4$ metric (29) span a moduli space

$$\mathcal{M} = \mathbb{R}_{>0}^1 \times \mathbb{R}_{>0}^1 \times \mathcal{M}_{\text{AHS}} \times \mathcal{M}_{\text{ACS}}. \quad (74)$$

To see that the metric on this moduli space coincides with the coset metric for the \mathcal{M}_{AHS} and \mathcal{M}_{ACS} factors, we proceed as follows. Let us assume that the metric on moduli space agrees with the moduli space metric from naive dimensional reduction, discussed in Sec. 3.5.2 of Ref. [35],

$$ds_{\mathcal{M}, \text{Naive}}^2 = \frac{2}{3} \left(\frac{\delta V_6}{V_6} \right)^2 + 2 \int_{\mathcal{X}_{m,n}} d^6 x \sqrt{\tilde{G}} \left(\frac{1}{4} \tilde{G}^{mp} \tilde{G}^{nq} \delta \tilde{G}_{mn} \delta \tilde{G}_{pq} - \frac{1}{4} (\tilde{G}^{mn} \delta \tilde{G}_{mn})^2 \right), \quad (75)$$

where $V_6 = V_4 V_2$. Here, \tilde{G} is the 6D unit metric

$$d\tilde{s}^2 = (V_6)^{-1/3} \left[(2V_4)^{1/2} d\tilde{s}_4^2 + V_2 d\tilde{s}_2^2 \right], \quad (76)$$

in terms of the 4D and 2D unit metrics on \mathcal{F}_4^* and \mathcal{F}_2^* ,

$$d\tilde{s}_4^2 = \bar{G}_{mn} dx^m dx^n = \Delta^{-1} Z G_{\alpha\beta} dx^\alpha dx^\beta + \Delta Z^{-1} (dx^4 + A)^2, \quad (77)$$

$$d\tilde{s}_2^2 = \frac{1}{\text{Im } \tau_2} |\eta^1 + \tau_2 \eta^2|^2. \quad (78)$$

Due to the \mathbb{Z}_2 identification under the involution (31), the 6D and 4D unit metrics have volume 1/2 in our conventions, and their \mathbb{Z}_2 covering spaces have volume 1.

By steps closely analogous to those in Ref. [35], the naive moduli space metric can be evaluated, giving

$$ds_{\mathcal{M}, \text{Naive}}^2 = \frac{2}{3} \left(\frac{\delta V_6}{V_6} \right)^2 + \left(\frac{\delta(V_4^{1/2} V_6^{-1/3})}{V_4^{1/2} V_6^{-1/3}} \right)^2 + \left(\frac{\delta(V_2 V_6^{-1/3})}{(V_2 V_6^{-1/3})} \right)^2 + ds_{\text{AHS}}^2 + ds_{\text{ACS}}^2,$$

which simplifies to

$$ds_{\mathcal{M}, \text{Naive}}^2 = \frac{2}{3} \left(\frac{\delta V_6}{V_6} \right)^2 + \frac{1}{3} \left(\frac{\delta V_{\text{rel}}}{V_{\text{rel}}} \right)^2 + ds_{\text{AHS}}^2 + ds_{\text{ACS}}^2. \quad (79)$$

Here, $V_{\text{rel}} = V_2/(V_4)^{1/2}$ and the last two terms are the natural coset metrics on \mathcal{M}_{AHS} and \mathcal{M}_{ACS} ,

$$ds_{\text{AHS}}^2 = \frac{1}{4} G_{\alpha\gamma} G_{\beta\delta} (\delta G^{\alpha\beta} dG^{\gamma\delta} + \tilde{\delta}\beta^{\alpha\beta} \tilde{\delta}\beta^{\gamma\delta}) + \delta_{IJ} G_{\alpha\beta} \delta x^{I\alpha} \delta x^{J\beta}, \quad (80a)$$

$$ds_{\text{ACS}}^2 = \frac{1}{2} \left| \frac{\delta \tau_2}{\text{Im } \tau_2} \right|^2, \quad (80b)$$

where

$$\tilde{\delta}\beta^{\alpha\beta} = \delta\beta^{\alpha\beta} - x^{I\alpha} \delta x^{I\beta} + x^{I\beta} \delta x^{I\alpha}. \quad (81)$$

For each choice of metric moduli, we have a unique triple j^α from Sec. 4.4 and unique w^1, w^2 from Sec. 4.2. Together, these define a unique $SU(2)$ structure on $\mathcal{X}_{m,n}$ with $SU(2)$ invariant tensors (13).

5 First-order Calabi-Yau metric and moduli space

5.1 Metric

Below the scale at which $\mathcal{N} = 4$ is spontaneously broken to $\mathcal{N} = 2$, the tree level type IIB supergravity description of T^6/\mathbb{Z}_2 with flux [26, 32, 33] is dual to IIA compactified on $\mathcal{X}_{m,n}$ with “first-order” metric [34]

$$ds_6^2 = Z \left(\frac{2s}{\text{Im } \tau_2} |dx^1 + \tau_2 dx^2|^2 + \bar{n} h \text{Im } \tau_1 (dx^3)^2 \right) + Z^{-1} \frac{\bar{n} h}{\text{Im } \tau_1} (\eta^4)^2 + \frac{\bar{m} h}{\text{Im } \tau_2} |\eta^1 + \tau_2 \eta^2|^2, \quad (82)$$

which is a restricted form of the $\mathcal{N} = 4$ metric (29). Here, $(\bar{m}, \bar{n}) = (m, n)/\gcd(m, n)$ and we identify $\text{Re } \tau_1$ with $-\beta^{12}$. The quantities β^{23}, β^{31} are fixed, as we will see below, and are no longer moduli. Compared to Eq. (29), we have

$$\begin{aligned} \Delta^{-1} G_{\alpha\beta} &= \frac{1}{\text{Im } \tau_2} |dx^1 + \tau_2 dx^2|^2 + \frac{\bar{n}h}{2s} (dx^3)^2, \\ 2V_4 &= (\bar{n}h)(2s), \quad V_2 = \bar{m}h, \quad \Delta = \sqrt{\frac{\bar{n}h}{2s}} \frac{1}{\text{Im } \tau_1}, \quad \beta^{12} = -\text{Re } \tau_2. \end{aligned} \quad (83)$$

Since the curvatures $d\eta^1 = dA^1$, $d\eta^2 = dA^2$ and $d\eta^4 = dA$ all vanish when restricted to constant x^1, x^2 , this metric describes a fibration whose generic fibers at constant x^1, x^2 are tori $T^4_{\{y^1, y^2, x^3, x^4\}}$. Due to the \mathbb{Z}_2 involution (31), the base is $\mathbb{P}^1 = T^2_{\{x^1, x^2\}}/\mathbb{Z}_2$. This is also borne out in the analysis of Refs. [34] and [12], which show that $\mathcal{X}_{m,n}$ is special type of T^4 fibration with projective embeddings, known as an abelian surface fibration over \mathbb{P}^1 . On the region $Z > 0$, the first-order metric (82) is a positive definite Calabi-Yau metric. It has not only $SU(2)$ structure, but $SU(3)$ holonomy. The harmonic Kähler form and holomorphic 3-form are

$$J = j^3 + w^1 \wedge w^2 = h(\bar{m}\eta^1 \wedge \eta^1 + \bar{n}dx^3 \wedge \eta^4) + 2sZdx^1 \wedge dx^2, \quad (84a)$$

$$\Omega = i(j^1 + ij^2) \wedge (w^1 + iw^2) = N(dx^1 + \tau_2 dx^2) \wedge (\eta^4 - \text{Im } \tau_1 Z dx^3) \wedge (\eta^1 + \tau_2 \eta^2), \quad (84b)$$

where

$$N = \sqrt{\frac{2V_6}{\text{Im } \tau_1 (\text{Im } \tau_2)^2}}. \quad (85)$$

It is straightforward to check that $dJ = d\Omega = 0$, and that

$$*J = \frac{1}{2} J \wedge J, \quad *\Omega = i\bar{\Omega}, \quad (86)$$

so that J and Ω are also co-closed. The normalization factor N ensures that

$$V_6 = \frac{1}{6} \int_{\mathcal{X}_{m,n}} J \wedge J \wedge J = \frac{i}{8} \int_{\mathcal{X}_{m,n}} \Omega \wedge \bar{\Omega}. \quad (87)$$

Finally, we write $\Omega = N\Omega_{\text{hol}}$, where Ω_{hol} depends on the complex structure moduli purely holomorphically. The Kähler and complex structure moduli are h, s, x^{I3} and $\tau_1, \tau_2, x^{I1} + \tau_2 x^{I2}$, respectively, for a total of $2 + M$ of each. The Hodge numbers $h^{11} = h^{21} = M + 2$ can also be deduced from the number of massless vector and hyper multiplets in the dual T^6/\mathbb{Z}_2 orientifold.

5.2 Moduli space of Calabi-Yau metrics on $\mathcal{X}_{m,n}$

The moduli space metric of the previous section restricts to the moduli space of the first-order Calabi-Yau metric to give

$$ds^2_{\mathcal{N}=2} = \frac{1}{2} \left(\frac{\delta V_6}{V_6} \right)^2 + 2ds^2_{\text{Kähler}} + 2ds^2_{\text{Complex}}, \quad (88)$$

where the Kähler and complex structure moduli spaces are given by

$$4ds_{\text{Kähler}}^2 = 2\left(\frac{\delta s_2}{s_2}\right)^2 + \left(\frac{\delta s_1}{s_1}\right)^2 + \frac{2}{s_1 s_2} \sum_{I=1}^M (s_2 \delta x^{I3})^2, \quad (89a)$$

$$4ds_{\text{Complex}}^2 = 2\left|\frac{\delta \tau_2}{\tau_2}\right|^2 + \left|\frac{\tilde{\delta} \tau_1}{\tau_1}\right|^2 + \frac{2}{\tau_1 \tau_2} \sum_{I=1}^M |\delta x^{I1} + \tau_2 \delta x^{I2}|^2. \quad (89b)$$

Here,

$$s_1 = 2s, \quad s_2 = \bar{n}h, \quad \text{and} \quad \tilde{\delta} \tau_1 = \delta \tau_1 + x^{I1} \delta x^{I2} - x^{I2} \delta x^{I1}. \quad (90)$$

To obtain this Calabi-Yau moduli space metric, we substitute the restricted choice (83) into the SU(2) structure moduli space metric (79), and drop the terms from $\tilde{\delta} \beta^{23}$ and $\tilde{\delta} \beta^{31}$.

The first term in Eq. (88) is as expected. The physical moduli space metric for the overall volume modulus from dimensional reduction differs from the standard Calabi-Yau moduli space metric by this term. Note that the moduli space metric of $\mathcal{X}_{m,n}$ is self mirror. The Kähler and complex structure moduli space metrics take the same form up to complexification. At $\text{Re } \tau_1 = \text{Re } \tau_2 = x^{I1} = 0$, we have

$$ds_{\text{Kähler}}^2 \leftrightarrow ds_{\text{Complex}}^2 \quad \text{under} \quad (v_1, v_2, x^{I3}) \leftrightarrow (\text{Im } \tau_1, \text{Im } \tau_2, x^{I2}). \quad (91)$$

The variables appearing in Eq. (88) are not the standard Kähler and complex structure moduli of $\mathcal{X}_{m,n}$, but are related by the nonlinear transformation (94) below. From Eqs. (20), (56) and (66), the nontwisted cohomology class of J is

$$\begin{aligned} [J] &= \frac{1}{2} s_2 [\xi^3 - C^{3\alpha} \xi_\alpha + 2x^{I3} \xi^I + \frac{m}{n} \eta^1 \wedge \eta^2] + \frac{1}{2} s_1 [\xi_3] + s_2, \\ &= \frac{1}{2} s_2 [\xi^3 + \frac{m}{n} \eta^1 \wedge \eta^2 - C^{31} \xi_1 - C^{32} \xi_2] + \frac{1}{2} (s_1 + s_2 x^{I3} x^{I3}) [\xi_3] + s_2 x^{I3} [\xi_I], \end{aligned} \quad (92)$$

where $C^{3\alpha} = \beta^{3\alpha} - x^{I3} x^{I\alpha}$. Here, the forms $s_2 \xi_1 = h d\eta^2$ and $s_1 \xi_2 = -h d\eta^1$ are exact. Dropping exact terms, we have

$$[J] = \frac{1}{2} v^2 [\xi^3 + \frac{m}{n} \eta^1 \wedge \eta^2] + \frac{1}{2} v^1 [\xi_3] + v^I [\xi_I], \quad (93)$$

where

$$v^2 = s_2, \quad v^1 = s_1 + s_2 x^{I3} x^{I3}, \quad v^I = s_2 x^I. \quad (94)$$

This allows us to compute the Calabi-Yau volume

$$V_6 = \frac{1}{6} \int_{\mathcal{X}_{m,n}} J \wedge J \wedge J = \frac{1}{2} s_1 s_2^2 = \frac{1}{6} \frac{m}{n} C_{ABC} v^A v^B v^C, \quad (95)$$

where

$$\frac{1}{6} C_{ABC} v^A v^B v^C = \frac{1}{2} v^2 (v^1 v^2 - v^I v^I). \quad (96)$$

Starting from the Kähler potential $K_{\text{Kähler}}$,

$$\exp(-K_{\text{Kähler}}) = 8V_6 = \frac{1}{4} \frac{m}{n} C_{ABC} v^A v^B v^C, \quad (97)$$

one can compute the Kähler moduli space metric $\partial_A \bar{\partial}_B K_{\text{Kähler}} \delta v^A \delta v^B$ and show via Eq. (94) that we indeed reproduce the metric (89a). This is done in App. A.

A similar statement holds for the complex structure moduli space. Writing

$$\text{Im } u^2 = \text{Im } \tau_2, \quad \text{Im } u^1 = \text{Im } \tau_1 + \text{Im } \tau_2 x^{I2} x^{I2}, \quad \text{Im } u^I = \text{Im } \tau_2 x^{2I}, \quad (98)$$

the complex structure volume is

$$\begin{aligned} V_{\text{cpx}} &= \frac{i}{8} \int_{\mathcal{X}_{m,n}} \Omega_{\text{hol}} \wedge \bar{\Omega}_{\text{hol}} \\ &= \frac{1}{2} \text{Im } \tau_1 (\text{Im } \tau_2)^2 = \frac{i}{6} C_{ABC} \left(\frac{u^A - \bar{u}^A}{2} \right) \wedge \left(\frac{u^B - \bar{u}^B}{2} \right) \wedge \left(\frac{u^C - \bar{u}^C}{2} \right), \end{aligned} \quad (99)$$

for the same C_{ABC} . Starting from the Kähler potential K_{cpx} ,

$$\exp(-K_{\text{cpx}}) = 8V_{\text{cpx}}, \quad (100)$$

one can compute the complex structure moduli space metric $\partial_A \bar{\partial}_B K_{\text{cpx}}(u, \bar{u}) \delta u^A \delta \bar{u}^B$ and show via Eq. (98) that we also reproduce Eq. (89b).

The holomorphic 3-form has nontwisted cohomology class

$$\Omega_{\text{hol}} = \left[\left((\xi^1 + \tau_2 \xi^2) + \tau_1 (\xi_2 - \tau_2 \xi_1) + (x^{I1} + \tau_2 x^{I2}) (2\xi_I + x^{I1} \xi_1 + x^{I2} \xi_2 + (1+\alpha) x^{I3} \xi_3) \right) \wedge (\eta^1 + \tau_2 \eta^2) \right]. \quad (101)$$

Here, we have fixed the massive $\mathcal{N} = 4$ scalars β^{13} and β^{23} to take the values

$$\beta^{13} = \alpha x^{I1} x^{I3} \quad \text{and} \quad \beta^{23} = \alpha x^{I2} x^{I3}. \quad (102)$$

We now show that $\alpha = -1$. Recall that the Kähler and complex structure moduli spaces only locally form a product. Globally, the Kähler moduli space can be fibered over the complex structure moduli space, but not vice versa, since the order of logic in constructing a Calabi-Yau metric is as follows:

1. Choose a complex structure on \mathcal{X} .
2. Choose a real Kähler class $[J] \in H^{1,1}(X, \mathbb{C})$ of positive norm.
3. By Yau's theorem, there exists a unique Ricci flat metric on X , with Kähler form in this cohomology class.

Therefore, the holomorphic 3-form cannot depend on the Kähler modulus x^{I3} , and we conclude that $\alpha = -1$. Alternatively, the form $\xi^3 \wedge (\eta^1 + \tau_2 \eta^2)$ appearing in Eq. (101), with coefficient proportional to $(1 + \alpha)$, is not closed:

$$d\left(\frac{1}{2} \xi^3 \wedge (\eta^1 + \tau_2 \eta^2)\right) = m \eta^1 \wedge \eta^2 \wedge (\xi^2 - \tau_2 \xi^3). \quad (103)$$

For $d\Omega_{\text{hol}} = 0$, we require that $\alpha = -1$.

5.3 Harmonic forms

It is convenient to express the quantity Z as

$$Z = 1 + \left(\frac{\bar{n}h}{2s}\right)\hat{Z}, \quad (104)$$

where \hat{Z} satisfies the rescaled Poisson equation

$$\left(\partial_3^2 + \frac{\bar{n}h \operatorname{Im} \tau_1}{2s} \nabla_{\hat{T}^2}^2\right)\hat{Z} = \sum_{\text{sources } s} Q_s(\delta^3(\mathbf{x} - \mathbf{x}^I) - 1), \quad \int d^3x \hat{Z} = 0. \quad (105)$$

Here \hat{T}^2 is $T_{x^1x^2}^2$ with unit area,

$$ds_{\hat{T}^2}^2 = \frac{1}{\operatorname{Im} \tau_2} |dx^1 + \tau_2 dx^2|^2. \quad (106)$$

Expressing the Kähler form (84a) in terms of \hat{Z} , we have [34]

$$J = s\omega_A + h\omega_H, \quad (107)$$

where

$$\omega_H = \bar{m}\eta^1 \wedge \eta^2 + \bar{n}\hat{Z}dx^1 \wedge dx^2 + \bar{n}dx^3 \wedge \eta^4 + \frac{h}{s}d\lambda, \quad (108a)$$

$$\omega_A = 2dx^1 \wedge dx^2 - d\lambda, \quad (108b)$$

and we choose λ so that ω_H and ω_S are harmonic. As in Sec. 4.4, we also have harmonic forms ω_I , for $I = 1, \dots, M$.

In the first-order description (29), $\omega_H, \omega_S, \omega_I$ form a basis for $H^2(\mathcal{X}_{m,n}, \mathbb{R})$. Letting \mathcal{E}_I denote the Poincaré dual of ω_I , we find the following intersection numbers $A \cdot B \cdot C = \int \omega_A \wedge \omega_B \wedge \omega_C$:

$$H^2 \cdot A = 2\bar{m}\bar{n}, \quad H \cdot \mathcal{E}_I \cdot \mathcal{E}_J = -\bar{m}\delta_{IJ}, \quad \text{others} = 0. \quad (109)$$

The forms $\omega_H, \omega_A, \omega_I$ are real and moduli dependent. From Eq. (93), we have

$$[\omega_H] = [\bar{m}\eta^1 \wedge \eta^2 + \frac{1}{2}\bar{n}\zeta^3] \quad \text{and} \quad [\omega_A] = [\zeta_3], \quad (110)$$

and it is convenient to define moduli independent forms

$$\xi_H = \omega_H \Big|_{\beta = x = 0} = \bar{m}\eta^1 \wedge \eta^2 + \frac{1}{2}\bar{n}\xi^3 \quad \text{and} \quad \xi_A = \omega_A \Big|_{\beta = x = 0} = \xi_3, \quad (111)$$

with ξ_I defined as in Eq. (65). In Sec. 6.4, we will relate the $\xi_a = (\xi_H, \xi_A, \xi_I)$ to integral classes on $H_4(\mathcal{X}_{m,n}, \mathbb{Z})$. We denote the dual Poincaré dual homology classes by $(H^0, A^0, \mathcal{E}_I^0)$. The form η_H restricted to an abelian fiber A gives the Hodge form $\bar{m}dy^1 \wedge dy^2 + \bar{n}dx^3 \wedge dx^4$ of the abelian surface fiber. As we will see in Sec. 6.4.3, the class $H \in H_4(\mathcal{X}_{m,n})$ is closely related to (but not quite the same as) the classes of Hodge surfaces of the abelian fibration.

6 Twisted versus ordinary Calabi-Yau cohomology

6.1 Interpretation in terms of supersymmetry breaking

In this section we consider two differential operators on $\mathcal{X}_{m,n}$: the ordinary exterior derivative and the twisted exterior derivative including torsion. The breaking of $\mathcal{N} = 4$ to $\mathcal{N} = 2$ is conveniently summarized in the lifting of some of the cohomology classes of the latter, so that twisted cohomology ring $H_{\text{tw}}^\bullet(\mathcal{X}_{m,n}, \mathbb{Z})$ is larger than the ordinary untwisted cohomology ring $H^\bullet(\mathcal{X}_{m,n}, \mathbb{Z})$. The former is closely related to the $SU(2)$ structure and underlying $\mathcal{N} = 4$ supersymmetry, and the latter to the standard description of a Calabi-Yau 3-fold as a manifold of $SU(3)$ holonomy preserving unbroken $\mathcal{N} = 2$ supersymmetry. Equivalently, some of the zero modes of the twisted Laplacian on $\mathcal{X}_{m,n}$ becomes nonzero (massive) modes of the ordinary Laplacian.¹¹

Let us first consider the de Rham cohomology, which includes only the free part of the integer cohomology and sets to zero the torsion classes. (Note that two different meanings of the word “torsion” used here. The distinction should be clear from context.) From field counting alone we deduce the lifting

$$\begin{aligned} \dim(H_{\text{tw}}^2(\mathcal{X}_{m,n}, \mathbb{R})) = M + 7 &\rightarrow \dim(H^2(\mathcal{X}_{m,n}, \mathbb{R})) = h^{11} = M + 2, \\ \dim(H_{\text{tw}}^3(\mathcal{X}_{m,n}, \mathbb{R})) = 2(M + 6) &\rightarrow \dim(H^3(\mathcal{X}_{m,n}, \mathbb{R})) = 2(h^{21} + 1) = 2(M + 3). \end{aligned} \quad (112)$$

In terms of homology, the complete statement is

$$\begin{aligned} H_0^{\text{tw}}(\mathbb{R}) &= \langle 1 \text{ cycle} \rangle, \\ H_1^{\text{tw}}(\mathbb{R}) &= \langle 2 \text{ boundaries} \rangle, \\ H_2^{\text{tw}}(\mathbb{R}) &= \langle 2 \text{ non-closed chains, } 3 \text{ boundaries, } (M + 2) \text{ cycles} \rangle, \\ H_3^{\text{tw}}(\mathbb{R}) &= \langle 3 \text{ non-closed chains, } 3 \text{ boundaries, } 2(M + 3) \text{ cycles} \rangle, \\ H_4^{\text{tw}}(\mathbb{R}) &= \langle 3 \text{ non-closed chains, } 3 \text{ boundaries, } (M + 2) \text{ cycles} \rangle, \\ H_5^{\text{tw}}(\mathbb{R}) &= \langle 2 \text{ non-closed chains} \rangle, \\ H_6^{\text{tw}}(\mathbb{R}) &= \langle 1 \text{ cycle} \rangle. \end{aligned} \quad (113)$$

Here, angle brackets mean “is generated by,” and the right hand side is the characterization based on the ordinary non-twisted exterior derivative. The word “boundaries” refers to non-twisted cycles that are trivial in real homology; in integer homology, they are the torsion cycles of Eqs. (24) and (26).

6.2 Twisted cohomology

A convenient of basis of global 1-forms and 2-forms that generate the twisted cohomology ring consists of

$$\text{1-forms } \eta^1, \eta^2 \quad \text{and} \quad \text{2-forms } \xi^m, \xi_m, \xi_I \quad (114)$$

¹¹In fact, the particular breaking of $\mathcal{N} = 4$ to $\mathcal{N} = 2$ here is special in that the two gravitino masses are equal [33]. This uniquely determines the form of the superhiggs mechanism, and all mass eigenvalues in terms of a single gravitino mass.

of Secs. 4.4 and 4.5, where $m = 1, 2, 3$, and $I = 1, \dots, M$. (Recall that $M = 16 - 4mn$.) Here, the existence of η^1 and η^2 is implied by the $SU(2)$ structure, and the 2-forms ξ_a arise from deformations of the triple of $SU(2)$ invariant 2-forms. The ξ_a satisfy

$$\xi^1 \wedge \xi_1 = \xi^2 \wedge \xi_2 = \xi^3 \wedge \xi_3 \equiv 2\gamma \quad \text{and} \quad \xi_I \wedge \xi_J \cong -\gamma \delta_{IJ} \quad \text{in cohomology,} \quad (115)$$

with other products of 2-forms vanishing. Upon integrating, we have $\int \eta^1 \wedge \eta^2 \wedge \gamma = 1$. Since the generators η^ρ and ξ_a are twisted-closed, their products generate the twisted cohomology ring:

$$\begin{aligned} H_{\text{tw}}^0(\mathbb{R}) &= \langle 1 \rangle \quad (\dim = 1), \\ H_{\text{tw}}^1(\mathbb{R}) &= \langle \eta^1, \eta^2 \rangle \quad (\dim = 2), \\ H_{\text{tw}}^2(\mathbb{R}) &= \langle \eta^1 \wedge \eta^2, \xi^m, \xi_m, \xi_I \rangle \quad (\dim = M + 7), \\ H_{\text{tw}}^3(\mathbb{R}) &= \langle \eta^1 \wedge \xi^m, \eta^1 \wedge \xi_m, \eta^1 \wedge \xi_I, \eta^2 \wedge \xi_m, \eta^2 \wedge \xi^m, \eta^2 \wedge \xi_I \rangle \quad (\dim = 2(M + 6)), \\ H_{\text{tw}}^4(\mathbb{R}) &= \langle \gamma, \eta^1 \wedge \eta^2 \wedge \xi_m, \eta^1 \wedge \eta^2 \wedge \xi^m, \eta^1 \wedge \eta^2 \wedge \xi_I \rangle \quad (\dim = M + 7), \\ H_{\text{tw}}^5(\mathbb{R}) &= \langle \gamma \wedge \eta^2, \gamma \wedge \eta^1 \rangle \quad (\dim = 2) \\ H_{\text{tw}}^6(\mathbb{R}) &= \langle \eta^1 \wedge \eta^2 \wedge \gamma \rangle \quad (\dim = 1). \end{aligned} \quad (116)$$

These forms have explicit expressions in the first-order metric (cf. Secs. 4.1 and 4.4).

6.3 Recovery of standard Calabi-Yau cohomology

To see which of these forms is lifted in ordinary (non twisted) cohomology, we need only the closure conditions on the generating 1-forms and 2-forms. These are:

$$\begin{aligned} d\eta^1 &= -n\xi_2, \quad d\eta^2 = n\xi_1 \\ \frac{1}{2}d\xi^1 &= -m\eta^1 \wedge \xi_3, \quad \frac{1}{2}d\xi^2 = -m\eta^2 \wedge \xi_3, \quad \frac{1}{2}d\xi^3 = m(\eta^1 \wedge \xi_1 + \eta^2 \wedge \xi_2), \end{aligned} \quad (117)$$

with $d\xi_m = d\xi_I = 0$. Note that the first line implies

$$d(\eta^1 \wedge \eta^2) = -n(\eta^1 \wedge \xi_1 + \eta^2 \wedge \xi_2), \quad (118)$$

so that the form $\eta^1 \wedge \xi_1 + \eta^2 \wedge \xi_2$ multiplied by either m or n is trivial in integer cohomology. Therefore, the same form multiplied by $\gcd(m, n)$ is trivial in integer cohomology, and $\eta^1 \wedge \xi_1 + \eta^2 \wedge \xi_2$ generates a $\mathbb{Z}_{\gcd(m, n)}$ torsion class. Note also that the linear combination $\xi_H = \frac{1}{2}\bar{n}\xi^3 + \bar{m}\eta^1 \wedge \eta^2$ is closed, where $(\bar{m}, \bar{n}) = \frac{1}{\gcd(m, n)}(m, n)$.

For the closure conditions on 3-forms, the nontrivial equations are

$$\begin{aligned} \frac{1}{2}d(\eta^2 \wedge \xi^1) &= (n\gamma - m\eta^1 \wedge \eta^2 \wedge \xi_3) = \gcd(m, n)(\bar{n}\gamma - \bar{m}\eta^1 \wedge \eta^2 \wedge \xi_3), \\ \frac{1}{2}d(\eta^1 \wedge \xi^2) &= -(n\gamma - m\eta^1 \wedge \eta^2 \wedge \xi_3) = -\gcd(m, n)(\bar{n}\gamma - \bar{m}\eta^1 \wedge \eta^2 \wedge \xi_3), \\ \frac{1}{2}d(\eta^1 \wedge \xi^3) &= -m\eta^1 \wedge \eta^2 \wedge \xi_2, \\ \frac{1}{2}d(\eta^2 \wedge \xi^3) &= m\eta^1 \wedge \eta^2 \wedge \xi_1, \end{aligned} \quad (119)$$

with $d(\eta^1 \wedge \xi_m) = d(\eta^2 \wedge \xi_m) = 0$.

For the closure conditions on 4-forms, the nontrivial equations are.

$$\begin{aligned}\frac{1}{2}d(\eta^1 \wedge \eta^2 \wedge \xi^1) &= -n\eta^1 \wedge \gamma, \\ \frac{1}{2}d(\eta^1 \wedge \eta^2 \wedge \xi^2) &= -n\eta^3 \wedge \gamma.\end{aligned}\tag{120}$$

In the equations above with a $\frac{1}{2}$ on the left hand side, the 2-form being differentiated is not integral, but differs from an integral form by a closed form. Therefore, the right hand side is indeed exact in integer cohomology. In summary, the free non-twisted cohomology groups are

$$\begin{aligned}H_{\text{free}}^0 &= \langle 1 \rangle = \mathbb{Z}, \\ H_{\text{free}}^1 &= \emptyset, \\ H_{\text{free}}^2 &= \langle \xi_3, \xi_H, \xi_I \rangle = \mathbb{Z}^{M+2}, \\ H_{\text{free}}^3 &= \langle w^\rho \wedge \xi_1, w^\rho \wedge \xi_2, w^1 \wedge \xi^1, w^2 \wedge \xi^2, w^1 \wedge \xi^2 + w^2 \wedge \xi^1, w^\rho \wedge \xi_I \rangle / \langle w^1 \wedge \xi_1 + w^2 \wedge \xi_2 \rangle = \mathbb{Z}^{2(M+3)}, \\ H_{\text{free}}^4 &= \langle w^1 \wedge w^2 \wedge \xi_H, w^1 \wedge w^2 \wedge \xi_3, \gamma, w^1 \wedge w^2 \wedge \xi_I \rangle / \langle \bar{n}\gamma - \bar{m}w^1 \wedge w^2 \wedge \xi_3 \rangle = \mathbb{Z}^{M+2}, \\ H_{\text{free}}^5 &= \emptyset, \\ H_{\text{free}}^6 &= \langle \eta^1 \wedge \eta^2 \wedge \gamma \rangle = \mathbb{Z},\end{aligned}\tag{121}$$

and

$$\begin{aligned}H_{\text{tor}}^0 &= \emptyset, \\ H_{\text{tor}}^1 &= \emptyset, \\ H_{\text{tor}}^2 &= \langle \xi_1, \xi_2, \rangle = \mathbb{Z}_n \times \mathbb{Z}_n, \\ H_{\text{tor}}^3 &= \langle \eta^1 \wedge \xi_3, \eta^2 \wedge \xi_3, \eta^1 \wedge \xi_1 + \eta^2 \wedge \xi_2 \rangle = \mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_{\text{gcd}(m,n)}, \\ H_{\text{tor}}^4 &= \langle \eta^1 \wedge \eta^2 \wedge \xi_1, \eta^1 \wedge \eta^2 \wedge \xi_2, \bar{n}\gamma - \bar{m}\eta^1 \wedge \eta^2 \wedge \xi_3 \rangle = \mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_{\text{gcd}(m,n)}, \\ H_{\text{tor}}^5 &= \langle \eta^1 \wedge \gamma, \eta^2 \wedge \gamma \rangle = \mathbb{Z}_n \times \mathbb{Z}_n, \\ H_{\text{tor}}^6 &= \emptyset.\end{aligned}\tag{122}$$

up to a caveat explained in the discussion of integer cohomology the next section.

Here is another way to summarize the lifting of forms in going from twisted to ordinary cohomology:

1-forms: η^1, η^2 no longer closed.

2-forms: $n\xi_1, n\xi_2$ now exact,
 ξ^1, ξ^2 , one linear combination of ξ^3 and $\eta^1 \wedge \eta^2$ no longer closed.

3-forms: $m\eta^1 \wedge \xi_3, m\eta^2 \wedge \xi_3, \text{gcd}(m, n)(\eta^1 \wedge \xi_1 + \eta^2 \wedge \xi_2)$ now exact,
 $\eta^1 \wedge \xi^3, \eta^2 \wedge \xi^3$, one linear combination of $\eta^1 \wedge \xi^1$ and $\eta^2 \wedge \xi^2$ no longer closed.

4-forms: $m\eta^1 \wedge \eta^2 \wedge \xi_1, m\eta^1 \wedge \eta^2 \wedge \xi_2, \text{gcd}(m, n)(\bar{n}\gamma - \bar{m}\eta^1 \wedge \eta^2 \wedge \xi_3)$ now exact,
 $\eta^1 \wedge \eta^2 \wedge \xi^1, \eta^1 \wedge \eta^2 \wedge \xi^2$ no longer closed.

5-forms: $n\eta^1 \wedge \gamma, n\eta^2 \wedge \gamma$ now exact.

This supplies the non-closed chains and boundary cycles required by Eq. (113).

6.4 The integer homology of $\mathcal{X}_{m,n}$

In this section we discuss the integer homology ring of the Calabi-Yau threefold $\mathcal{X}_{m,n}$ and its relation to the homology basis $S^0, H^0, \mathcal{E}_I^0$ of Eq. (22).

6.4.1 Hodge surfaces of $\mathcal{X}_{m,n}$

Abelian surfaces and Hodge curves

An abelian surface is a projective variety that has the additional structure of the abelian group $U(1)^4$. That is,

$$A \cong \mathbb{C}^4 / \Lambda \cong T^4, \quad (123)$$

with addition of points defined as in \mathbb{C}^4 . For A to be embeddable in a projective space, it must have Kähler class proportional to an integer Hodge form [22]. If we choose coordinates y^1, y^2, x^3, x^4 on A with periodicity $y^\rho \cong y^\rho + 1$ and $x^\rho \cong x^\rho + 1$, then an integer form $m dy^1 \wedge dy^2 + n dx^3 \wedge dx^4$ is proportional to

$$\omega = \bar{m} dy^1 \wedge dy^2 + \bar{n} dx^3 \wedge dx^4, \quad (124)$$

where bars denote division by $\gcd(m, n)$. For an abelian surface, at most one of \bar{m} and \bar{n} can differ from one. By Poincaré duality, we can view $[\omega]$ as an element of either $H^2(A, \mathbb{Z})$ or $H_2(A, \mathbb{Z})$, and we will use the same notation for both. If $[\omega]$ is represented by an irreducible Hodge curve $C \subset A$, then, since $c_1(A) = 0$, the genus of this curve is determined by its Euler characteristic

$$2g - 2 = C \cdot C = \int \omega \wedge \omega = 2\bar{m}\bar{n} \int_{T^4} dy^1 \wedge dy^2 \wedge dx^3 \wedge dx^4 = 2\bar{m}\bar{n}, \quad (125)$$

to be $g = \bar{m}\bar{n} + 1$.

Abelian surface fibrations and Hodge surfaces

To say that the Calabi-Yau manifold $\mathcal{X}_{m,n}$ is an abelian surface fibration over \mathbb{P}^1 means that all of the structure of the preceding paragraph is fibered over \mathbb{P}^1 . A Hodge class $[\omega] \in H^2(\mathcal{X}_{m,n}, \mathbb{Z}) \cong H_4(\mathcal{X}_{m,n}, \mathbb{Z})$ is represented by a Hodge surface S , which is itself fibered by genus $g = \bar{m}\bar{n}$ curves C over \mathbb{P}^1 .

Given an embedding $\iota: S \hookrightarrow \mathcal{X}_{m,n}$ of the type we have just described, the projection $\pi: \mathcal{X}_{m,n} \rightarrow \mathbb{P}^1$ induces a projection $\pi' = \pi \circ \iota: S \rightarrow \mathbb{P}^1$ with base $\pi'(S) = \mathbb{P}^1$ and generic fiber $\pi'^{-1}(p) \cong C$ for $p \in \mathbb{P}^1$. Then, every section ℓ of S determines a section $\sigma = \iota(\ell)$ of $\mathcal{X}_{m,n}$, since $\ell \subset S$ meets each fiber A in a single point $p \in C \subset A$.

An abelian surface fibration comes with a zero section, and with the operation of addition of sections defined. Given the zero section σ_0 , another section σ of $\mathcal{X}_{m,n}$, and an embedding $S \hookrightarrow \mathcal{X}_{m,n}$, we can define a second embedding by shifting abelian fiber coordinates by $\sigma - \sigma_0$.

The Calabi-Yau threefold $\mathcal{X}_{m,n}$.

Recall from above that $\mathcal{X}_{m,n}$ has $b_4 = M + 2$, where $M = 16 - 4mn$, with possible values 12, 8, 4, 0. The $M + 2$ homology classes can be realized as follows.

For $M \neq 0$, each Hodge surface S has $2M$ sections ℓ_I, ℓ'_I , for $I = 1, \dots, M$, with homology classes satisfying $[\ell_I] + [\ell'_I] = [C'] \in H_2(S, \mathbb{Z})$, independent of I . Given an embedding $\iota: S \hookrightarrow \mathcal{X}_{m,n}$, the $2M$ sections ℓ_I and ℓ'_I of S determine $2M$ embeddings ι_I and $\iota'_I: S \hookrightarrow \mathcal{X}_{m,n}$. One of these coincides with the original embedding, provided that one of the sections of $\iota(S)$, denoted S_0 coincides with the zero section σ_0 of $\mathcal{X}_{m,n}$. We focus on this case, and write $S_I = \iota_I(S)$ and $S'_I = \iota'_I(S)$, with homology classes satisfying $[S_I] + [S'_I] = [D] \in H_4(\mathcal{X}_{m,n}, \mathbb{Z})$, independent of I . Then, $H_4^{\text{free}}(\mathcal{X}_{m,n}, \mathbb{Z})$ is generated by the $M + 1$ independent classes from the Hodge surfaces, together with the class of the Abelian surface fiber $[A]$.

For $M = 0$, each Hodge surface S has a single section. Given an embedding $\iota: S \hookrightarrow \mathcal{X}_{m,n}$, this section of S determines another embedding $\iota_0: S \hookrightarrow \mathcal{X}_{m,n}$. The two embeddings coincide, provided the zero section of $\iota(S)$ coincides with the zero section σ_0 of $\mathcal{X}_{m,n}$. Again, we focus on this case, and write $S_0 = \iota(S)$. Then $H_4^{\text{free}}(\mathcal{X}_{m,n}, \mathbb{Z})$ is generated by $[S_0]$ together with the class of the Abelian surface fiber $[A]$.

6.4.2 The Mordell-Weil lattice D_M

The sections σ of the abelian surface fibration $\mathcal{X}_{m,n}$ form a lattice known as the Mordell-Weil lattice $\text{MW}(\mathcal{X}_{m,n}) / \text{MW}^{\text{tor}}(\mathcal{X}_{m,n})$. From the group addition law, we already know that we can add any two sections to obtain a new section. To define a lattice, we also need an inner product between sections, which in this context is known as a “height pairing.” Since the sections are curves, they do not generically intersect in the threefold $\mathcal{X}_{m,n}$, but do in a surface, and this is exactly the additional structure we have at our disposal. We can compute intersections in a Hodge surface of the abelian fibration.

As discussed above, a Hodge surface S of $\mathcal{X}_{m,n}$ is a fibered over \mathbb{P}^1 by curves C of genus $g = \bar{m}\bar{n}$, and has $2M$ sections ℓ_I and ℓ'_I , for $I = 1, \dots, M$, satisfying $\ell_I + \ell'_I = C'$, with C' independent of I . The intersections of these curves in S are

$$\ell_I^2 = \ell'_I{}^2 = -\bar{m}, \quad \ell_I \cdot \ell'_I = \bar{m}, \quad \text{and} \quad \ell_I \cdot \ell_J = \ell'_I \cdot \ell_J = \ell'_I \cdot \ell'_J = 0 \quad \text{for } I \neq J, \quad (126)$$

together with $\ell \cdot C = 1$ and $C^2 = 2\bar{m}\bar{n}$. We assume that we are given an embedding $\iota: S \hookrightarrow \mathcal{X}_{m,n}$ such that one of the sections, denoted ℓ_0 , maps to the zero section of $\mathcal{X}_{m,n}$: $\iota(\ell_0) = \sigma_0$. Then, $H_2(S, \mathbb{Z})$ is spanned by the $M + 1$ independent sections, together with the genus $\bar{m}\bar{n}$ curve C , and their images under ι likewise span $H_2(\mathcal{X}_{m,n}, \mathbb{Z})$ up to torsion classes. Thus, for any two sections of $\mathcal{X}_{m,n}$, we can compute the intersection of the corresponding homology classes in S . A height pairing on sections of $\mathcal{X}_{m,n}$ is given by the intersection pairing on the orthogonal complement $\langle \ell, C \rangle^\perp$ in S .

On S , let us choose $\ell_0 = \ell'_M$. Then, the projection to $\langle \ell, C \rangle^\perp$ in $H_2(S, \mathbb{Z})$ maps the

sections of S as

$$\begin{aligned}
\ell_I &\mapsto \ell_I^\perp = \ell_I - \ell'_M - C, \\
\ell'_I &\mapsto \ell'^\perp_I = \ell'_I - \ell'_M - C, \quad I = 1, \dots, M, \\
\ell_M &\mapsto \ell_M^\perp = \ell_M - \ell'_M - 2C, \\
\ell'_M &\mapsto \ell'^\perp_M = 0.
\end{aligned} \tag{127}$$

The lattice $\langle \ell, C \rangle^\perp$ is $-\bar{m}(D_M)$, where (D_M) denotes the root lattice of D_M the prefactor denotes $-\bar{m}$ times the usual Cartan inner product. If we choose D_M roots $a_I = [\ell_I^\perp - \ell'^\perp_{I+1}]$ for $I = 1, \dots, M-1$ and $a_M = -[\ell_{M-1}^\perp]$, then we have

$$\begin{aligned}
a_I &= [\ell'_I - \ell'_{I+1}], \\
a_{M-1} &= [\ell'_{M-1} - \ell'_M - C], \\
a_M &= [\ell'_{M-1} - \ell_M + C] \quad (\text{using } [-\ell_{M-1} + \ell'_M] = [\ell'_{M-1} - \ell_M]).
\end{aligned} \tag{128}$$

It is straightforward to check that the intersections of a_1, \dots, a_M as defined by Eq. (128) give $-\bar{m}$ times those of the D_M Dynkin diagram.

6.4.3 The relation to the basis of Sec. 5.3

We would now like to relate the cohomology classes of the Hodge surfaces S_I, S'_I and Abelian fiber A to those of the basis A, H^0, \mathcal{E}_I^0 defined in Sec. 5.3. From Eq. (109), we have

$$H \cdot \mathcal{E}_I^0 \cdot \mathcal{E}_J^0 = -\bar{m} \delta_{IJ}. \tag{129}$$

Therefore, the curves $e_I = H \cdot \mathcal{E}_I^0$ form an orthogonal basis on the surface H with normalization $e_I \cdot_H e_I = -\bar{m}$, in terms of which the roots generating a D_M lattice can be realized in the standard way,

$$e_1 - e_2, \quad e_2 - e_3, \quad \dots, \quad e_{M-1} - e_M, \quad e_{M-1} + e_M. \tag{130}$$

The only caveat is that H might not be a Hodge surface or even an integral cycle, and the “roots” constructed in this way might not be in $H_2(\mathcal{X}_{m,n}, \mathbb{Z})$. In fact, this will turn out to be the case. Nevertheless, it will be helpful to proceed in this way.

Identifying S with $S_0 = S'_M$, and using the intersections discussed in App. B below, Eq. (128) becomes

$$\begin{aligned}
a_I &= [\ell'_I + \ell'_M] - [\ell'_{I+1} + \ell'_M] = \frac{1}{\bar{m}\bar{n}} [S_I - S_{I+1}] \cdot [S], \\
a_{M-1} &= [\ell'_{M-1} + \ell'_M] - [2\ell'_M + C] = \frac{1}{\bar{m}\bar{n}} [S_{M-1} - S_M] \cdot [S], \\
a_I &= [\ell'_{M-1} + \ell'_M] - [\ell_M + \ell'_M - C] = \frac{1}{\bar{m}\bar{n}} [S_{M-1} - S'_M] \cdot [S],
\end{aligned} \tag{131}$$

or equivalently, using $[S_I + S'_I] = [D]$, independent of I ,

$$\begin{aligned} a_I &= \frac{1}{\bar{m}\bar{n}} \left[\frac{1}{2}(S_I - S'_I) - \frac{1}{2}(S_{I+1} - S'_{I+1}) \right] \cdot [S], \\ a_{M-1} &= \frac{1}{\bar{m}\bar{n}} \left[\frac{1}{2}(S_{M-1} - S'_{M-1}) - \frac{1}{2}(S_M - S'_M) \right] \cdot [S], \\ a_M &= \frac{1}{\bar{m}\bar{n}} \left[\frac{1}{2}(S_{M-1} - S'_{M-1}) + \frac{1}{2}(S_M - S'_M) \right] \cdot [S], \end{aligned} \quad (132)$$

so that

$$\begin{aligned} a_I &= [\mathcal{E}_I^0 - \mathcal{E}_{I+1}^0] \cdot [S], \\ a_{M-1} &= [\mathcal{E}_{M-1}^0 - \mathcal{E}_M^0] \cdot [S], \\ a_M &= [\mathcal{E}_{M-1}^0 + \mathcal{E}_M^0] \cdot [S] \end{aligned} \quad (133)$$

for

$$[\mathcal{E}_I^0] = \frac{1}{2\bar{m}\bar{n}} [S_I - S'_I]. \quad (134)$$

If H^0 coincides with a Hodge surface, then our identification of H^0, \mathcal{E}_I^0 is complete. In fact, H^0 does not coincide with a Hodge surface, but this line of reasoning will lead us to the correct identification.

To determine H let us start with the two defining properties $H^0 \cdot H^0 \cdot H^0 = 0$ and $H^0 \cdot H^0 \cdot A = 2\bar{m}\bar{n}$. The \mathcal{E}_I^0 as defined in Eq. (134), together with A and any one S_J , form a basis for $H_4^{\text{free}}(\mathcal{X}_{m,n}, \mathbb{Z})$. Of this basis, only S_J has nonzero intersection with A . From App. B, we have

$$S_J \cdot S_J \cdot A = 2\bar{m}\bar{n}. \quad (135)$$

Therefore, whether or not the \mathcal{E}_I^0 of Eq. (134) are the correct identification of \mathcal{E}_I^0 , H^0 has an expansion

$$[H] = [S_J] + c^a[A] + c^I[\mathcal{E}_I^0], \quad (136)$$

using the \mathcal{E}_I^0 so defined. By requiring that $H^3 = 0$, we find from the intersections of App. B a family of solutions for H^0 given by

$$[H^0] = [S_J] + \frac{2}{3}\bar{m}^2\bar{n}[A] + \text{shifts}, \quad (137)$$

where the “+ shifts” denotes shifts by an arbitrary linear combination of $[\bar{m}^2\bar{n}^2\mathcal{E}_I^0 + \frac{1}{2}\bar{m}^2\bar{n}A]$ for $I = 1 \dots, M$. In the case $M = 0$, there is a single S_0 , and none of these shifts exist. Therefore,

$$[H^0] = [S_0] + \frac{2}{3}\bar{m}^2\bar{n}[A], \quad \text{for } M = 0, \quad (138)$$

which is the case $(m, n) = (\bar{m}, \bar{n}) = (4, 1)$ or $(1, 4)$. Next, the from $\mathcal{E}_I^0 \cdot A = 0$, the \mathcal{E}_I^0 must either be as defined in Eq. (134), or must be a linear combination of the same classes. Either way, the condition $H^0 \cdot H^0 \cdot \mathcal{E}_I^0 = 0$ follows, for \mathcal{E}_I^0 as defined in Eq. (134). For $M \neq 0$, this uniquely fixes the shifts in Eq. (137). We find

$$[H] = \frac{1}{2}[S_J + S'_J] + \frac{\bar{m}^2\bar{n}}{6}[A] \quad \text{for } M \neq 0, \quad (139)$$

where the class $[S_J + S'_J] = [D]$ is independent of J . Eqs. (134), (138) and (139) complete our identification of the basis $[A], [H^0], [\mathcal{E}_I^0]$ of Sec. 5.3 in terms of the $[A]$ and the Hodge surfaces.

Thus, $[H^0]$ and $[\mathcal{E}^0]$ are not actually integer classes, but rather

$$\begin{aligned} [H^0] - \frac{\bar{m}^2 \bar{n}}{6} [A] &\in H_4(\mathcal{X}_{m,n}, \tfrac{1}{2}\mathbb{Z}) \quad (M \text{ arbitrary}), \\ \bar{m} \bar{n} [\mathcal{E}_I^0] &\in H_4(\mathcal{X}_{m,n}, \tfrac{1}{2}\mathbb{Z}) \quad \text{and} \quad [H^0] \pm \bar{m} \bar{n} [\mathcal{E}_I^0] \in H_4(\mathcal{X}_{m,n}, \mathbb{Z}) \quad (M \text{ nonzero}). \end{aligned} \quad (140)$$

Therefore, in the entries for $H_2^{\text{free}}(\mathcal{X}_{m,n}, \mathbb{Z})$ and $H_4^{\text{free}}(\mathcal{X}_{m,n}, \mathbb{Z})$, (121), ξ_H and ξ_I should be replaced by the linear combinations $\xi_H \pm \bar{m} \bar{n} \xi_I$ and $\bar{m} \bar{n} (\xi_I \pm \xi_H)$ for $M \neq 0$, and ξ_H should be replaced by $2\xi_H$ for $M = 0$.

7 Conclusions

We have shown that the manifolds in class of abelian surface fibered Calabi-Yau threefolds $\mathcal{X}_{m,n}$ have a novel interpretation as manifolds of $SU(2)$ structure in addition to the conventional interpretation as manifolds of $SU(3)$ holonomy. The threefolds are obtained from duality to the type IIB T^6/\mathbb{Z}_2 orientifold, in which the choice of flux spontaneously breaks $\mathcal{N} = 4$ to $\mathcal{N} = 2$ supersymmetry. In the $SU(2)$ structure description, the topology of the Calabi-Yau topology spontaneously break $\mathcal{N} = 4$ to $\mathcal{N} = 2$ supersymmetry at a scale that can be made hierarchically larger than the compactification scale by taking the base to be large.

As a manifestation of the $SU(2)$ structure, the frame bundle of $\mathcal{X}_{m,n}$ splits as the sum of 4D and 2D subbundles, and the moduli space of $SU(2)$ structure metrics enlarge the Calabi-Yau metric moduli space, with a natural interpretation in terms of spaces of almost hypercomplex structure and almost complex structure on the 4D and 2D bundles, respectively. The analysis was facilitated by the existence of an explicit family of metrics approximating the exact $SU(2)$ structure and Calabi-Yau metrics, whose harmonic forms can be written down, and which yields exact topological information.

We were able to compute the metric on Calabi-Yau metric moduli space and show that the restriction from the $SU(2)$ structure moduli space metric agrees with the Calabi-Yau moduli space metric computed in the conventional way from the special geometry determined by classical triple intersection numbers of $\mathcal{X}_{m,n}$. The light scalars of the $\mathcal{N} = 4$ theory were determined by a twisted cohomology ring associated with the $SU(2)$ structure, of which only the exact Calabi-Yau moduli of the $\mathcal{N} = 2$ theory remain in the standard cohomology ring. For nonminimal topological data $(m, n) \neq (1, 1)$, the lifted cohomology classes of the twisted cohomology ring persist as torsion classes of the standard cohomology ring.

Finally, to be able to provide precise statements in integer homology, we have extended the results of Ref. [12] on the integer homology ring of $\mathcal{X}_{m,n}$. This has allowed us to relate a basis of cohomology classes from the first-order analysis of harmonic forms to exact integer homology classes of $\mathcal{X}_{m,n}$ based on embeddings of Hodge surfaces of $\mathcal{X}_{m,n}$.

The dual T^6/\mathbb{Z}_2 orientifold with $\mathcal{N} = 2$ flux is a simple model of a type IIB warped compactification, embodying features of more realistic models. In future work, we plan to apply the results of this investigation to provide a derivation of the procedure for warped Kaluza-Klein reduction of type IIB string theory on T^6/\mathbb{Z}_2 by duality to standard compactification of type IIA string theory on the class of Calabi-Yau manifolds $\mathcal{X}_{m,n}$. Nearly all string compactifications of phenomenological interest are warped, yet explicit examples of warped Kaluza-Klein reduction are almost nonexistent, despite a promising proposal for a general framework in which to understand them [36, 14, 16, 17, 41], building on earlier work [11, 19]. We hope to provide a complementary approach to Refs. [36, 14, 16], and to provide explicit examples with which to probe that formalism.

Another interesting line of research concerns connections between the $\mathcal{X}_{m,n}$ for different (m, n) . By going to a point in complex structure modulus at which $\text{MW}(\mathcal{X}_{1,1})$ develops \mathbb{Z}_m torsion, it appears possible to resolve the singular manifold in such a way that the new principally polarized abelian fibration has $\pi_1 = \mathbb{Z}_m$. Similar π_1 changing topological transitions have been considered in Refs. [8, 9]. The transition relate $\mathcal{X}_{1,1}$ to a new manifold that lies midway between $\mathcal{X}_{m,1}$ and $\mathcal{X}_{1,m}$ in the sense that lifting to an m -fold cover gives the former and quotienting by \mathbb{Z}_m gives the latter. These transitions are currently under investigation [13]. The manifolds $\mathcal{X}_{m,n}$ are of potential interest in heterotic model building since they have very few moduli and fundamental groups $\mathbb{Z}_n \times \mathbb{Z}_n$ useful for Wilson lines. It is worth understanding transitions of this sort, which could yield other new manifolds of phenomenological interest.

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A Kähler moduli space metric

In this Appendix, we show that the Kähler potential (97) indeed leads to the metric (89a) on Kähler moduli space. Let us complexify the v^A of Sec. 5.2 by writing $t^A = b^A + iv^A$. Then, the Kähler potential (97) becomes

$$\begin{aligned} K &= -\log 8V = -\log \frac{i}{3!} \frac{m}{n} C_{ABC} (t^A - \bar{t}^A)(t^B - \bar{t}^B)(t^C - \bar{t}^C) \\ &= -\log \frac{m}{n} (t^2 - \bar{t}^2) ((t^1 - \bar{t}^1)(t^2 - \bar{t}^2) - (\mathbf{t} - \bar{\mathbf{t}})^2) \\ &= -\log(t^2 - \bar{t}^2) - \log((t^1 - \bar{t}^1)(t^2 - \bar{t}^2) - (\mathbf{t} - \bar{\mathbf{t}})^2) + \text{const.} \end{aligned} \tag{141}$$

Differentiating gives Kähler metric $K_{A\bar{B}} = \partial_A \bar{\partial}_B K$, with components

$$\begin{aligned}
4K_{1\bar{1}} &= \frac{(v^2)^2}{(v^1 v^2 - \mathbf{v}^2)^2}, \\
4K_{2\bar{2}} &= \frac{1}{(v^2)^2} + \frac{(v^1)^2}{(v^1 v^2 - \mathbf{v}^2)^2}, \\
4K_{1\bar{2}} &= -\frac{1}{v^1 v^2 - \mathbf{v}^2} + \frac{v^1 v^2}{(v^1 v^2 - \mathbf{v}^2)^2}, \\
4K_{I\bar{J}} &= \frac{2\delta_{IJ}}{v^1 v^2 - \mathbf{v}^2} + \frac{4v^I v^J}{(v^1 v^2 - \mathbf{v}^2)^2}, \\
4K_{1\bar{I}} &= -\frac{2v^I v^2}{(v^1 v^2 - \mathbf{v}^2)^2}, \\
4K_{2\bar{I}} &= -\frac{2v^I v^1}{(v^1 v^2 - \mathbf{v}^2)^2},
\end{aligned} \tag{142}$$

and $K_{B\bar{A}} = K_{A\bar{B}}$. Then, restricting to $t^A = iv^A$ purely imaginary, we find

$$\begin{aligned}
4ds_{\text{Kähler}}^2 &= 4K_{A\bar{B}} \delta v^A \delta v^B \\
&= \left(\frac{\delta v^2}{v^2} \right)^2 + \frac{1}{(v^1 v^2 - \mathbf{v}^2)^2} (\delta(v^1 v^2 - \mathbf{v}^2))^2 - \frac{2}{v^1 v^2 - \mathbf{v}^2} (\delta v^1 \delta v^2 - (\delta \mathbf{v})^2).
\end{aligned} \tag{143}$$

If we introduce new variables s_1, s_2, x^{3I} via

$$\begin{aligned}
v^2 &= s_2, \\
v^1 &= s_1 - s_2 \delta_{IJ} x^{3I} x^{3J}, \\
v^I &= s_2 x^{3I},
\end{aligned} \tag{144}$$

then $v^1 v^2 - \mathbf{v}^2 = s_1 s_2$, and it is possible to show that

$$(\delta \mathbf{v})^2 - \delta v^1 \delta v^2 = (s_2)^2 (\delta \mathbf{x}^3)^2 - \delta s_1 \delta s_2. \tag{145}$$

Therefore, the previous expression for the Kähler metric becomes

$$\begin{aligned}
4ds_{\text{Kähler}}^2 &= \left(\frac{\delta s_2}{s_2} \right)^2 + \left(\frac{\delta(s_1 s_2)}{s_1 s_2} \right)^2 - \frac{2}{s_1 s_2} (s_2 (\delta \mathbf{x}^3)^2 - \delta s_1 \delta s_2) \\
&= 2 \left(\frac{\delta s_2}{s_2} \right)^2 + \left(\frac{\delta s_1}{s_1} \right)^2 - \frac{2}{s_1 s_2} (s_2 (\delta \mathbf{x}^3)^2),
\end{aligned} \tag{146}$$

as claimed in Sec. 5.2.

B Intersections of Hodge surfaces in $\mathcal{X}_{m,n}$

In this Appendix, we compute the double and triple intersections of integer divisors in the Calabi-Yau manifolds $\mathcal{X}_{m,n}$, extending the results of Ref. [12].

For the case $(m, n) = (1, 1)$, the Calabi-Yau manifold $\mathcal{X}_{1,1}$ was realized in the second construction of Ref. [12] as the relative Jacobian $\text{Pic}^0(S/\mathbb{P}^1)$ of a surface S fibered by genus-2 curves over \mathbb{P}^1 . In this case, the Hodge curves and surfaces of Secs. 6.4.1 are known as theta curves and theta surfaces.

As explained Sec. 4.3.3 and App. J of Ref. [12], the sections of S can be viewed as sections of $\text{Pic}^1(S/\mathbb{P}^1)$, while those of $\mathcal{X}_{1,1}$ can be viewed as sections of $\text{Pic}^0(S/\mathbb{P}^1)$. To relate the two, we note that $\text{Pic}^0(S/\mathbb{P}^1) \cong \text{Pic}^0(S/\mathbb{P}^1)$ under a noncanonical isomorphism obtained by tensoring with a privileged section of S ,

$$\text{Pic}^1(S/\mathbb{P}^1) \xrightarrow{\otimes[\ell_0]^{-1}} \text{Pic}^0(S/\mathbb{P}^1), \quad \ell \mapsto \ell - \ell_0, \quad (147)$$

where $\ell_0 \in \{\ell_I, \ell'_I\}$. The isomorphism depends on the choice of which of the $2M = 24$ sections ℓ_I, ℓ'_I of S maps to the zero section of \mathcal{X} .

Similarly, each section of $\text{Pic}^1(S/\mathbb{P}^1)$ determines a theta surface, embedding S into $\text{Pic}^0(S/\mathbb{P}^1)$. Thus, for the $2M$ choices $\ell_0 = \ell_I$ and ℓ'_I , for $I = 1, \dots, M$, we obtain $2M$ theta surfaces Θ_I and Θ'_I in $\mathcal{X}_{1,1} \cong \text{Pic}^0(S/\mathbb{P}^1)$. App. J of Ref. [12] computed the intersections of these theta surfaces to be

$$\begin{aligned} \Theta_I \cdot \Theta_J &= \sigma_0 + \sigma_{\ell'_I - \ell_J}, & \Theta'_I \cdot \Theta'_J &= \sigma_0 + \sigma_{\ell_I - \ell'_J}, \\ \Theta'_I \cdot \Theta_J &= \sigma_0 + \sigma_{\ell_I - \ell_J}, & \Theta_I \cdot \Theta'_I &= 2\sigma_0 + C_{\ell_I \cap \ell'_I}. \end{aligned} \quad (148)$$

Here, $L \rightarrow \sigma_L$ is the isomorphism identifying degree zero line bundles in $\text{Pic}^0(S/\mathbb{P}^1)$ with sections of the abelian fibered threefold \mathcal{X} . The curve $C_{\ell_I \cap \ell'_I}$ is the common genus-2 fiber of Θ_I and Θ'_I . Note that $\ell'_I - \ell_J = \ell'_J - \ell_I$ as a consequence of

$$[\ell_I + \ell'_I] = [C'], \quad \text{independent of } I. \quad (149)$$

For self intersections, we have

$$[\Theta_I \cdot \Theta_I] = c_1(K_{\Theta_I}), \quad (150)$$

where for the surface S , it was shown that

$$c_1(K_S) = [C'] - [C]. \quad (151)$$

Here, $[C]$ is the class of the genus-2 fiber of S . The triple intersections were shown to be

$$\begin{aligned} \Theta_I \cdot \Theta_J \cdot \Theta_K &= 1, & A \cdot \Theta_I \cdot \Theta_J &= A \cdot \Theta_I \cdot \Theta'_J = 2, \\ \Theta_I \cdot \Theta_J \cdot \Theta_J &= \Theta_I \cdot \Theta'_J \cdot \Theta'_J = -2, & A \cdot \Theta_I \cdot \Theta_J &= 2, \\ \Theta_I \cdot \Theta_I \cdot \Theta'_I &= \Theta_I \cdot \Theta_J \cdot \Theta'_J = 0, & A \cdot \Theta_I \cdot \Theta_J &= 2, \\ \Theta_I \cdot \Theta_I \cdot \Theta_I &= -4, & A \cdot \Theta_I \cdot \Theta_I &= 2. \end{aligned} \quad (152)$$

together with equations obtained from these by exchange of Θ and Θ' .

Double intersections in $\mathcal{X}_{m,n}$

For general (m, n) we claim that the generalization of Eq. (148) for intersections in $\mathcal{X}_{m,n}$ is

$$\begin{aligned} [S_I \cdot S_J] &= \bar{m}\bar{n}[\sigma_0] + \bar{m}\bar{n}[\sigma_{\ell'_I - \ell_J}], & [S'_I \cdot S'_J] &= \bar{m}\bar{n}[\sigma_0] + \bar{m}\bar{n}[\sigma_{\ell_I - \ell'_J}], \\ [S'_I \cdot S_J] &= \bar{m}\bar{n}[\sigma_0] + \bar{m}\bar{n}[\sigma_{\ell_I - \ell_J}], & [S_I \cdot S'_I] &= 2\bar{m}\bar{n}[\sigma_0] + \bar{m}^2\bar{n}[C_{\ell_I \cap \ell'_I}], \end{aligned} \quad (153)$$

with self intersections given by

$$[S_I \cdot S_I] = c_1(K_{S_I}), \quad \text{with} \quad K_S = \bar{m}\bar{n}[C'] - \bar{m}^2\bar{n}[C]. \quad (154)$$

That two Hodge surfaces generically intersect in $2\bar{m}\bar{n}$ sections is clear fiberwise, since in each T^4 fiber A , two genus- $(1 + \bar{m}\bar{n})$ curves C generically intersect $2\bar{m}\bar{n}$ points (cf. Eq. (125)). On the other hand the subscripts in Eq. (153) require explanation. Since S is fibered by genus $\bar{m}\bar{n}$ curves and $\text{Pic}^0 S/\mathbb{P}^1$ is a $T^{2+2\bar{m}\bar{n}}$ fibration over \mathbb{P}^1 , it is not immediately clear why sections of $\mathcal{X}_{m,n}$ should be associated with sections of $\text{Pic}^0(S/\mathbb{P}^1)$. Let us focus on the case $(m, n) = (1, n)$. In this case, we have a map $f: \text{Pic}^0(S/\mathbb{P}^1) \rightarrow \mathcal{X}_{1,n}$ acting fiberwise as

$$(x^1, x^2; y^{1i}, y^{2i}) \mapsto (x^1, x^2, y^1, y^2) = (x^1, x^2; \sum_{i=1}^n y^{1i}, \sum_{i=1}^n y^{2i}). \quad (155)$$

The inverse map f^{-1} gives an embedding of an n^2 -fold cover of $\mathcal{X}_{1,n}$ into $\text{Pic}^0(S/\mathbb{P}^1)$,

$$(x^1, x^2; y^1, y^2) \mapsto (x^1, x^2, y^{1i}, y^{2i}), \quad \text{with} \quad y^{1i} = y^1/n, \quad y^{2i} = y^2/n. \quad (156)$$

This map takes the principally polarized T^{2+2n} fibration $\text{Pic}^0(S/\mathbb{P}^1)$ to the $(n, 1)$ polarized T^4 fibration $\mathcal{X}_{1,2}$.

Let us show that the polarizations are as claimed. Under the inverse map, the Kähler form

$$J_{\text{fib}} = h(dx^1 \wedge dx^2 + \sum_{i=1}^n dy^{1i} \wedge dy^{2i}) \quad (157)$$

on the T^{2+n} fibers of $\text{Pic}^0(S/\mathbb{P}^1)$ pulls back to Kähler form

$$(f^{-1})^* J_{\text{fib}} = h(dx^1 \wedge dx^2 + \sum_{i=1}^n \frac{dy^1}{n} \wedge \frac{dy^2}{n}) = h'(ndx^1 \wedge dx^2 + dy^1 \wedge dy^2) \quad (158)$$

on the T^4 fibers of $\mathcal{X}_{1,n}$, where $h' = h/n$. Alternatively, since $\mathcal{X}_{1,n} = \mathcal{X}_{n,1}/(\mathbb{Z}_n \times \mathbb{Z}_n)$, we can view $(\hat{y}^1, \hat{y}^2) = (y^1/n, y^2/n)$ as coordinates on the fibers of $\mathcal{X}_{n,1}$, and the previous pullback gives fiber Kähler form

$$h(dx^1 \wedge dx^2 + n d\hat{y}^1 \wedge d\hat{y}^2) \quad (159)$$

on the fibers of $\mathcal{X}_{n,1}$. The remaining case $(m, n) = (2, 2)$ is obtained as $\mathcal{X}_{2,2} = \mathcal{X}_{4,1}/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Triple intersections in $\mathcal{X}_{m,n}$

The triple intersections can be obtained as double intersections of curves in surfaces. For example, for I, J, K distinct,

$$\begin{aligned}
S_I \cdot S_J \cdot S_K &= (S_I \cdot S_J) \cdot_{S_J} (S_J \cdot S_K) \\
&= (\bar{m}\bar{n}[\sigma_0] + \bar{m}\bar{n}[\sigma_{\ell'_I - \ell_J}]) \cdot_{S_J} (\bar{m}\bar{n}[\sigma_0] + \bar{m}\bar{n}[\sigma_{\ell'_K - \ell_J}]) \\
&\cong \bar{m}^2 \bar{n}^2 (\ell_J + \ell'_I) \cdot_S (\ell_J + \ell'_K) \\
&= -\bar{m}^3 \bar{n}^2.
\end{aligned} \tag{160}$$

Here, we have used the fact that $\sigma_{\ell - \ell_J}$ maps to $\ell \in S$ under the isomorphism $S_J \rightarrow S$. The remaining triple intersections of theta surfaces are

$$\begin{aligned}
S_I \cdot S_J \cdot S_J &= S_I \cdot S'_J \cdot S'_J = -2\bar{m}^3 \bar{n}^2, \\
S_I \cdot S_I \cdot S'_I &= S_I \cdot S_J \cdot S'_J = 0, \\
S_I \cdot S_I \cdot S_I &= -4\bar{m}^3 \bar{n}^2,
\end{aligned} \tag{161}$$

together with equations obtained from these by exchange of S and S' . The computation is analogous to the previous one.

From the independence the result on the choice of which of the three Hodge surfaces is used to perform the double intersection, we can deduce the coefficients of $[C]$ and $[C']$ in the double intersection expressions (153) and (154). For example, supposed that

$$[S_I \cdot S'_I] = 2\bar{m}\bar{n}[\sigma_0] + \lambda[C_{\ell_I \cap \ell'_I}]. \tag{162}$$

Then, for agreement of

$$\begin{aligned}
S_I \cdot S'_I \cdot S_J &= (S_I \cdot S'_I) \cdot_{S_I} (S_I \cdot S_J) \\
&= (2\bar{m}\bar{n}[\sigma_0] + \lambda[C_{\ell_I \cap \ell'_I}]) \cdot_{S_I} (\bar{m}\bar{n}[\sigma_0] + \bar{m}\bar{n}[\sigma_{\ell'_J - \ell_I}]) \\
&\cong \bar{m}^2 \bar{n}^2 (2\ell_I + \lambda[C]) \cdot_S (\ell_I + \ell'_J) \\
&= -2\bar{m}^3 \bar{n}^2 + 2\lambda\bar{m}\bar{n},
\end{aligned} \tag{163}$$

and

$$\begin{aligned}
S_I \cdot S'_I \cdot S_J &= (S_I \cdot S_J) \cdot_{S_J} (S'_I \cdot S_J) \\
&= (\bar{m}\bar{n}[\sigma_0] + \bar{m}\bar{n}[\sigma_{\ell'_I - \ell_J}]) \cdot_{S_J} (\bar{m}\bar{n}[\sigma_0] + \bar{m}\bar{n}[\sigma_{\ell_I - \ell_J}]) \\
&\cong \bar{m}^2 \bar{n}^2 (\ell_J + \ell'_I) \cdot_S (\ell_J + \ell_I) \\
&= 0,
\end{aligned} \tag{164}$$

we require $\lambda = \bar{m}^2 \bar{n}$. Similarly, writing $K_S = \alpha[C'] - \beta[C]$, we find from

$$S_I \cdot S_I \cdot S_J = (S_I \cdot S_I) \cdot_{S_I} (S_I \cdot S_J) = (S_I \cdot S_J) \cdot_{S_J} (S_I \cdot S_J) \tag{165}$$

that $\beta = \bar{m}^2 \bar{n}$, and then from

$$S_I \cdot S_I \cdot S'_I = (S_I \cdot S_I) \cdot_{S_I} (S_I \cdot S'_I) = (S_I \cdot S'_I) \cdot_{S'_I} (S_I \cdot S'_I) \tag{166}$$

that $\alpha = \bar{m}\bar{n}$.

Finally, let us compute the triple intersections involving the generic abelian surface fiber A . In this case, $A^2 = 0$, and

$$A \cdot S_I \cdot S_J = A \cdot S_I \cdot S'_J = A \cdot S'_I \cdot S'_J = 2\bar{m}\bar{n}, \quad (167)$$

for any I, J , not necessarily distinct. This is most easily proven from the intersection of curves in the abelian fiber A . For example,

$$\begin{aligned} A \cdot S_I \cdot S_J &= (A \cdot S_I) \cdot_A (A \cdot S_J) \\ &\cong C \cdot_A C = 2\bar{m}\bar{n}, \end{aligned} \quad (168)$$

as desired. (In an abelian surface, the self-intersection of a genus- g curve is $2g - 2$, and we have $g = 1 + \bar{m}\bar{n}$.)

The same result is obtained if the intersections are performed in a theta surface. Let C_I denote the genus-2 fiber of $S_I \cong S$. Then, for example, for $I \neq J$,

$$\begin{aligned} A \cdot S_I \cdot S_J &= (A \cdot S_I) \cdot_{S_I} (S_I \cdot S_J) \\ &= C_I \cdot_{S_I} (\bar{m}\bar{n}[\sigma_0] + \bar{m}\bar{n}[\sigma_{\ell'_J - \ell_I}]) \\ &\cong C \cdot_S (\bar{m}\bar{n}[\ell_I] + \bar{m}\bar{n}[\ell'_J]) = 2\bar{m}\bar{n}, \\ A \cdot S_I \cdot S_I &= (A \cdot S_I) \cdot_{S_I} [S_I \cdot S_I] \\ &= C_I \cdot_{S_I} [c_1(K_{S_I})] \\ &\cong [C] \cdot_S [\bar{m}\bar{n}C' - \bar{m}^2\bar{n}C] = 2\bar{m}\bar{n}. \end{aligned} \quad (169)$$

In the last step, we have used the fact that the genus $g = 1 + \bar{m}\bar{n}$ fiber C of S satisfies $C^2 = 0$ and $C \cdot C' = C \cdot (\ell_K + \ell'_K) = 2$, when the intersections are computed in S .

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